UNDERGRADUATE DIFFICULTIES:

ALGEBRAIC SKILLS AND MATHEMATICAL COMPREHENSION

BY T.M. BOUSTEAD

UNDERGRADUATE DIFFICULTIES:
ALGEBRAIC SKILLS AND MATHEMATICAL COMPREHENSION

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Therese Mary Boustead (nee Devonport)

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DECLARATION

The work presented in this thesis is, to the best of my knowledge and belief, original, except as acknowledged in the text. This material has not been submitted, either in whole or part, for a degree at this or any other University.

Therese Mary Boustead

April, 1999

ABSTRACT

Many first year university students struggle with mathematics. Observations in senior mathematics classes in four New Zealand secondary schools highlighted potential undergraduate problems, especially with algebraic and mathematical reading skills. In this thesis, these two areas are investigated further.

In the first part of the thesis, an analysis is done of algebraic tests given to first year university mathematics students. From the results there emerged five main categories of common consistent algebraic difficulties. These categories not only emerged the following year with a similar group, but senior secondary school and second year undergraduate mathematics students also displayed them. Overall, the conclusion was that these categories of algebraic difficulties formed from the research did not appear to improve with higher mathematical learning.

A second area for the investigation of undergraduate difficulties was in the field of reading to learn mathematics. The results of a questionnaire survey confirmed that students were not only resistant to reading mathematical text, but they did not appear to have the skills to read expository text. Many students used a narrative, surface approach to mathematical reading that resulted in very little of a topic being understood. Further analysis using a variety of extracts, case studies, interviews and written answers led to the formation of a mathematical reading model based on generative comprehension research by Wittrock (1990) and interactive reading research by Dechant (1991). For mathematical text, critical linkages were often symbol-symbol linkages requiring a higher level of comprehension than narrative text. These critical linkages were predominantly located at an inner text layer. A major deterrent to reading mathematical text for students is the difficulty in locating these critical linkages in hard-copy text. Further investigations compared hard copy text with various types of software designed for self-study purposes. Some of the software was found to be better at directing students to these critical linkages while others were not so successful.

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CONTENTS

Page

List of Figures

List of Tables

List of Tables continued

Chapter 1 Introduction

1.1 Secondary to tertiary transition

Students encounter significant epistomological/cognitive, social/cultural and didactical changes as they pass from school to university (de Guzman, Hodgson, Robert, & Villani, 1998). Although these changes have always existed for students entering university, it is only in relatively recent literature that wide ranging transition concerns have been documented, for example Billing, 1997; Boddy & Neale, 1998; Booth, 1997; Bradley & Kemp, 1993; Dearn, 1996; Demana, 1988; McInnis & Richard, 1995; Peel, 1996.

The change from secondary to tertiary study is a disorientation period, affected by earlier experiences and opposing environments.

> *What happens before is important (i.e. at school or college or employment), as is the considerable mismatch of school and higher education.* (Billing, 1997, p 132).

A mismatch occurs when:

- Small (< 30 students) interpersonal school classes change to large impersonal lectures (> 250 students).
- Regular help from teachers at school changes to intermittent help that the student has to actively seek at university.
- From school to university there is a decrease in class contact hours and an increase in self-study hours.
- There is a change in teaching methods and styles (Peel, 1996), often towards a more passive learning environment.

A common result of these changes is that students entering university have difficulty working out what is expected of them in terms of academic standards, behaviour and course requirements. Further research adds that success at university may be strongly influenced by the students' own attitudes and expectations (Boddy & Neale, 1998), how the students integrate socially into their new environment (Dearn, 1996; McInnis & Richard, 1995), background knowledge and how the students are inducted into the academic life (Booth, 1997).

Some research has focussed on the students' perceptions of their difficulties in the transition to university study. For example, in a 1996 survey of more than 900 students at Victoria University of Wellington, New Zealand, the students perceived that their greatest difficulty was *time management,* particularly as it related to assessment requirements. Time management should be a skill that can be taught at an early age and transferred from secondary to tertiary study.

Another study, but from the lecturer's perspective, indicates that although time management is important for university success an even more important skill is *independent learning* or *intellectual independence* (Dearn, 1996). Unlike time management, this is a non-transferable skill:

> *It appears that being able to learn independently is something that is developed over time within a particular domain.* (Dearn, p 194 -195).
>
> *It is only after a period of immersion in a field that the learner will be able to develop intellectual autonomy.* (Dearn, p. 201).

Dearn describes this skill as a climate for self direction, specific to a topic. The undergraduate experience can also be described as a cognitive apprenticeship as proposed by Collins, Brown and Newman, (1989). Over time students gradually acquire the language, values and way of thinking specific to a field. Intellectual autonomy is seen to form a significant part in the way students learn to think. This applies in all fields and is especially important in mathematics.

1.2 Mathematical transition

Researchers who were already looking at students' difficulties in mathematics have begun to focus more on the secondary/tertiary transition. These researcher include Bradley & Kemp, 1993; Cohen, 1982; Crawford, Gordon, & Nicholas, 1998; Dan, 1990; Demana, 1988; de Guzman et al., 1998; Zucker, 1996. For example, a questionnaire sent by de Guzman (1998) to various groups including engineering students, pre-service secondary teachers and mathematics students pointed to the following difficulties with the transition to university mathematics.

The students' perception included:

- Frustration with their own lack of skills in proofs and abstract developments.
- Acknowledgement of a lack of background prerequisite knowledge.
- Not being clear about course expectations (that is, what was essential in a topic and what was accessory).
- A lack of concrete examples.
- A lack of time allocated to classrooms.
- Many topics being covered too quickly.
- The idea that the lecturers' expectations of student knowledge were unrealistic.
- The fact that topics were presented once and not repeated as in schools.

The lecturers' perception included:

- A dissatisfaction with student weaknesses.
- A perception that students had a lack of interest in mathematics itself and were only interested in examination success.
- A lack of consolidating knowledge through self-study.

- An underestimation by students for the role of mathematics as a service in other topics.
- Disgust for the learning style used by students, particularly the preference for acquiring computational skills. (de Guzman, et al., 1998)

The lecturers' concerns therefore relate more to mathematical autonomy, that is, an awareness of mathematics, attitude to mathematics and study habits. Also included was a concern for weak background knowledge. Other research has cited these issues and found that strong influences on intellectual independence in mathematics are: previous knowledge affecting the hierarchical development of concepts (Schoenfeld, 1994), the students' conceptions of mathematics influencing their orientation to study (Crawford, Gordon, & Nicholas, 1998; Crawford, Gordon, Nicholas, & Prosser, 1994), and the way the students interpret content or relate to mathematical activities (Solomon, 1992).

A further difficulty specific to mathematics is the cognitive leap that occurs when there is a change from concepts that are intuitive and founded on experience, to formal definitions and properties constructed on logical deduction (Tall, 1997). This leap involves a dramatic change in perspective as well as an increase in topic depth, including technical difficulties associated with manipulating new objects and understanding what underlies them. Because of this leap many students find it difficult to obtain intellectual independence or autonomy in mathematics. A flow-on effect from the cognitive leap is that students may not know how to take notes in a mathematics class, they may not know how to plan to study mathematical topics and they may not know how to read a textbook (de Guzman et al., 1998).

1.3 Mathematics as an active construction

Throughout this study there is an underlying assumption that students actively construct their own knowledge.

In the field of cognitive science, Jean Piaget was the classic cognitive theorist who advocated active construction of knowledge, specifically in relation to mathematical thought processes (Inhelder & Piaget, 1958). Piaget outlined a developmental approach culminating with formal thinking. His stages of development began with a young child's concrete thought processes and progressed in a stepwise fashion to adolescent formal thinking. Piaget assumed that an individual's deductive reasoning and behaviour showed distinctive and consistent patterns. These ideas have since been challenged. For example, Karplus (1981) and his colleagues, categorised five different types of reasoning patterns. In contrast to Piaget's theory, the researchers also found that the same individual did not necessarily use consistent reasoning performance when confronted with different tasks. This has implications for the way students construct their mathematical knowledge, especially with self-study. The way one student would build mathematical knowledge would not necessarily be the same as another student. Also, the same student may approach different mathematical tasks in different ways.

Despite criticisms of Piaget's theory, his assertion that the learner actively constructed knowledge is relevant to this study. It is assumed that mathematics learning is more likely to be effective and longer lasting if the students are actively involved in trying to understand mathematical and abstract concepts (Ernest, 1994; Leder, 1993; Skemp, 1976; Steffe & Tzur, 1994; von Glaserfeld, 1994). This involves the active construction of meaning as an intellectual activity that uses higher order thinking skills such as evaluation, analysis, synthesis and reflection of existing knowledge, attitudes and values (Dearn, 1996). This active construction is aided by activities such as discussing, listening, reading, writing and reflecting.

1.4 Mathematics demonstrates extremes

How students learn actively has been the focus of cognitive research for some time. Throughout the literature researchers have discussed the difference between two extreme approaches. For example, Skemp (1976) distinguished between those who tried to understand why a relationship was true and when to apply it (relational understanding), and those who followed a set of rules to help pass assignments and examinations (instrumental understanding).

Phenomenographic research, that is, looking at different ways students relate to a phenomenon (via thinking, feeling and acting), highlights the difference between deep and surface learning (Marton & Saljo, 1976a; Marton & Saljo, 1976b; Ramsden, Beswich, & Bowden, 1987; Saljo, 1987). Marton and Saljo (1976a, 1976b) were leading investigators of active student approaches to learning. Their study used academic articles especially selected for their tight logical argument and lack of technical knowledge. They concluded that the way a student solved and reasoned through a problem was directly linked to how the student intended to learn concepts. Marton and Saljo identified four different types of responses ranging from a summary of the main argument supplemented with personal understanding, to a few isolated points with confusion or misunderstanding (Saljo, 1981). Two main independent approaches to learning were identified. With the *deep* approach the student intended to understand the meaning of the passage. With the *surface* approach the student intended to rote memorise parts of the passage. Marton found that students who adopted the *deep* approach had better detail recall after five weeks. The implication here was that the *deep* approach led to more effective understanding. However, work by (Svensson, 1977) found that a *deep* approach did not necessarily lead to a deep level of understanding if prior knowledge was inadequate. Svensson also found that it was not possible for the *surface* approach to lead to deep understanding. Students using a rote learning technique often found this process so tedious and unrewarding that they eventually did less and less work. Svensson noted that those students adopting the *surface* approach often ended the year by failing the examinations. This work was later confirmed by Entwistle and Ramsden when they sampled 2208 students across 66 academic departments (Entwistle & Ramsden, 1983). They added a third factor that included a student's motivation and called it *achieving*. This third factor described the combination of deep/surface learning approaches a student used to achieve an external goal, such as a university degree. The researchers recorded a clear link between approaches to study and the level of understanding and outcome, both in experimental and natural environments. They also found evidence that while students used the *deep* approach in both science and humanities, the emphasis on detail and procedures in science could successfully encourage the *surface* approach via rote learning while the emphasis on personal interpretation in the humanities encouraged *deep* understanding. They concluded that the discipline and method of teaching could influence the ways students tackled a task.

There have also been studies on the factors influencing the approach to learning. For example, it was easier to induce a *surface* approach to learning than a *deep* approach by adjusting the type of question asked (Marton and Saljo 1976a, 1976b). Students who took a *deep* approach easily adapted to surface-like questions while those who took a *surface* approach had great difficulty with *deep*-orientated questions. Likewise, students with anxiety, lack of interest or who perceived the topic to be irrelevant were less likely to adopt a *deep* approach (Fransson, 1977). On the other hand, high motivation and topic interest as perceived by the student, helped make the *deep* approach more likely to occur (Entwistle & Ramsden, 1983; Svensson, 1977).

Mathematics is a field that can clearly demonstrate extremes between relational and instrumental understanding or deep and surface approaches. There is a distinct difference between those students who concentrate on computation versus those who aim to understand the underlying meaning behind the computation. A significant challenge to first year university mathematics students is the emphasis on the use of advanced thinking and reasoning skills required to deal with more complex and abstract mathematical concepts. Relational understanding and deep approaches are expected. When confronting new knowledge, do the students try to make sense of new ideas using their current knowledge (natural learners), do they give new knowledge a chance to develop its own meaning before linking to other knowledge (formal learners), or are the students flexible enough to use both approaches whenever appropriate (Tall, Thomas, Davis, Gray, & Simpson, 1998)?

Research literature has therefore shown that students with significant gaps in background knowledge and with a *surface* approach to learning can find it difficult to adapt to the abstract concepts and advanced thinking skills that promote intellectual autonomy in mathematics.

Therefore this study has the following assumptions:

- Higher education should foster intellectual autonomy.
- Students actively construct their own knowledge.
- Active self-study and self-direction contributes to mathematical understanding.
- Prior knowledge contributes to mathematical success.
- Intellectual autonomy in mathematics is more attainable by adopting a deeper approach to learning, where students look for the meaning that underlies mathematical concepts.

1.5 The objective of this research

The knowledge, experience and background content developed in schools may have a significant impact on the development of intellectual autonomy at university. Although there has been research published on the changes from secondary to tertiary study and the difficulties associated with those changes, there are only pockets of research that give more depth to aspects that impact on mathematical autonomy. What is needed initially is a study of both current and potential impacts that are transferable from the schools directly into university mathematics. Next, some of the difficulties need to be looked at in greater depth. Are these difficulties addressed as mathematical autonomy is developed? Are there resources at the tertiary level that can be used to overcome the difficulties that slow or inhibit mathematical autonomy?

The purpose of the research then was to determine major influences from schools that contribute to the difficulties experienced by undergraduate mathematics students. The research is taken from the undergraduate perspective. It explores the difficulties undergraduate students have, especially in first year mathematics, as an inheritance from schools. Without actually solving the difficulties it points to possible ways to overcome the problems at both the senior secondary and undergraduate levels.

1.6 The research approach

The research approach was dictated by the questions that needed to be explored. At times questionnaires were used to obtain quantitative data while at other times the author deemed it more suitable to obtain depth by using qualitative techniques such as observations, interviews and case studies on small samples of students. The qualitative approach was intended to give direction and detail rather than inference to large cohorts. Often the quantitative approach was supplemented with a sample of short interviews. Most of the data was collected via questionnaires, written scripts, taped interviews and observations. In general, written consent was obtained from the students taking part in the research, although some general data was obtained from routine course work. Where possible validity was assured by either repeating tasks in consecutive years, or by collecting data from similar tasks.

Chapter 2: The research began by collecting data from senior classes in secondary schools. Observations and interviews highlighted a selection of potential impacts on tertiary mathematics. It was noticed that two issues, algebraic skills and mathematical reading of text, appeared to dominate the significant difficulties in undergraduate mathematics. A decision was made to pursue these two influences in more depth at the undergraduate level.

Chapters 3 and 4: *Algebraic skills.* At school, emphasis is placed on algebraic skills in early to mid secondary school, but the emphasis decreases at the middle to senior school level. A problem at university is that students cannot interpret solutions if they have not mastered enough algebraic skills to obtain a solution. Chapter 3 describes the analysis of two consecutive years of algebraic tests sat by first year mathematics students. The aim was to identify major areas of algebraic difficulties emerging from the schools. Data collected from the test results were supplemented with one-to-one interview sessions. In Chapter 4 this was taken a step further by determining whether higher learning had any effect on the algebraic skill difficulties categorised in Chapter 3. Chapter 4 finishes with a suggestion of a possible approach to remedy the difficulties at the university and senior school levels.

Chapters 5 to 9: *Reading to learn mathematics.*
Chapters 5 to 9 explore an area that is currently lacking in research literature: how students' reading of mathematical text is influenced by the quality of linkages in the text.

Students at the secondary level are not explicitly taught to read to learn mathematics and consequently this may cause considerable difficulty in first year mathematics at university. These chapters explore the phenomenon of reading mathematics. As a foundation for the study, Chapter 5 has a brief review of the literature on reading to learn. It introduces a possible theoretical basis for looking at mathematical reading, in particular Wittrock's generative comprehension model and Dechant's interactive model of reading (Dechant, 1991; Wittrock, 1990).

Initially, the aim was to collect as much information as possible on the extent to which mathematics is read by students and the processes they use. Chapter 6 describes the questionnaires that were given to all three levels of undergraduate students. Reading extracts were then given to first year students. These extracts were accompanied by a short questionnaire to determine both the processes and outcomes from the reading. This data was supplemented with in-depth taped interviews and one-to-one help sessions on a variety of first year mathematical topics.

From the data collected and analysed in Chapter 6 there emerged a picture of how students read mathematical text. As a foundation for explaining mathematical reading, Chapter 7 gives a breakdown of mathematical text that is built on comments and answers from student questionnaires. Chapter 7 then focuses on the possible linkages between text components and extends Wittrock's and Dechant's models into mathematical text comprehension.

With the rapid advancement of technology and the ease of access university students have to computers, it is inevitable that hard-copy text will be supplemented in the future by interactive packaged mathematical software. This software is reputed to present difficult mathematical topics in an easier, more user-friendly way. The aim of Chapter 8 was to look at a variety of different software available at the time of this research and to compare the comprehension of concepts with current hard-copy text. In Chapter 9, the success or failure of task comprehension with the software is examined in terms of the quality of linkages between mathematical text components.

Chapter 2 Impact from Schools

The various impacts from secondary to tertiary mathematics can involve a multi-dimensional array of psychological, pedagogical and epistemological influences. This part of the study looked at what was currently happening in the senior secondary school mathematics classrooms in New Zealand and what could impact on tertiary mathematics.

2.1 The New Zealand mathematics curriculum

The New Zealand school education system has undergone several significant changes in the last 40 years. In the late 1960's, a Euclidean geometry approach was replaced with an emphasis on sets, relationships and functions. In 1992, the Ministry of Education released 'Mathematics in the New Zealand Curriculum' (Ministry of Education, 1992), a document that once again changed the direction of teaching and learning of mathematics in the schools, for Years 1 to 13. This new curriculum was part of a broad initiative that concentrated on mathematical processes and was aimed at improving achievement levels. By specifying what students should be able to achieve (performance criteria) and regularly assessing that achievement, the purpose was to credit students with successful portions of study rather than an entire course of study. It also recommended that "...students participate in collaborative (real world) problem solving situations" (p.7) and assumed that "...both calculators and computers will be available and used in the teaching and learning of mathematics at all levels" (p.14). However, the continued existence of an external Bursary examination at the end of Year 13 also plays a role in determining a student's future, especially at university.

2.2 Schools selected for observation

A week was spent in each of four Christchurch city schools selected for their demographic and socioeconomic differences. Two of the schools were state, coeducational schools and two were single sex (one state boys' school and one private girls' school). The socioeconomic status (SES) of the schools sampled, rated by the Ministry of Education (1996) on a scale from 1 to 10, ranged from '3' (indicating a low socioeconomic level and the second lowest in Christchurch) to '10' (highest possible socioeconomic level). One of the coeducational schools (SES rating '3') was located in a less affluent section of the city while the other coeducational school (SES rating '9') was situated in a more affluent part of the city. Student attendance rolls for the schools ranged from just below 500 students to slightly more than 2000 students.

2.3 Observations and interviews

The impact of the secondary sector on undergraduate university mathematics was approached through observation of senior secondary mathematics classes and interviews with the teachers of those observed classes.

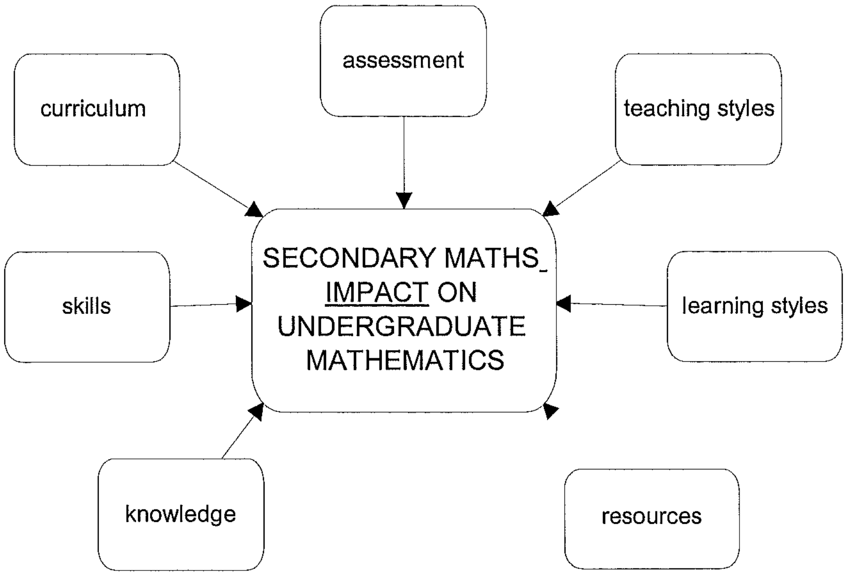


Figure 2.1: Diagram indicating possible factors that could impact on undergraduate university mathematics.

The research assumed that most of the impact on tertiary mathematics would come from senior levels of secondary mathematics. The diagram outlined in Figure 2.1 was based on this author's secondary and tertiary teaching experience and was used as a basis for both class observations and teacher interviews.

2.4 Classes observed

Observations were limited to senior mathematics classes, namely Years 12 and 13. Class sizes ranged from 15 to 32 students. Mathematics is an option taken by approximately 56% of the Year 12 secondary students. In Year 13, there are two courses students can choose from and each involves an end-of-year external examination. The Year 13 papers are part of a University Bursary Examination, a qualification designed to determine entry to university study. These two courses are 'Mathematics with Calculus' (taken by 35% of students sitting Bursary) and 'Mathematics with Statistics' (taken by 50% of students sitting Bursary) (Ministry of Education, 1996).

Students ranged between 15 years and 17 years of age. Observations were also centred on, but not exclusive to, those classes that were likely to have many of their students attending university the following year. Although 'top stream students' were in two of the observed classes, this streaming was not exclusive since class allocation also depended on student selection of other topics. In one school, only six students in a class of 15 students intended to enrol at university and all six students were interested in pursuing areas other than mathematics. At our local university, the initial focus for undergraduate mathematics centres on how the Year 13 students performed in the 'Mathematics with Calculus' examination. Therefore most observations for this study focussed on these classes rather than the Mathematics with Statistics classes. In total, observations took place in four Year 12 classes, one Year 13 Mathematics with Statistics class and ten Year 13 Mathematics with Calculus classes.

2.5 Interviews with teachers

Interviews with seven class teachers took place at the respective schools during non-contact teaching periods. The length of each interview varied between 30 minutes and 45 minutes. Teachers were asked specific questions on each of the factors outlined in the Figure 2.1. Conversations were taped and summary notes were taken during the interviews.

Table 2.1: Questions and answers from interviews with seven class teachers

Question 1	Is there anything that dictates what you teach?
Answers with number of similar replies in brackets	- The Bursary prescription (2) - Previous examination papers, then the Bursary prescription then the curriculum content (4) - Curriculum content only. Many of our students do not sit Bursary (1)
Question 2	What teaching style do you use most of the time?
	- I do examples on the board and build up the ideas by asking questions of the class (6) - Usually have students to work in pairs or together (1)
Question 3	What is your impression of the background mathematics knowledge of your students?
	- Not too bad. Algebra is a bit weak (6) - The students have trouble with problem solving (4) - Most have difficulty with the content in the 6th and 7th form (1)
Question 4	Do you encourage your students to use a textbook, and if so, what do you do?
	- The textbook is used mainly for doing exercises in class and at home (5) - The textbook is used to explain a topic. I get the students to read sections of it at times (2)
Question 5	Are computers used as part of the mathematics class, and if so, how are they used?
	- Computers are used only for selected topics such as functions and conics (2) - Computers are not used much. They are too inconvenient. One needs to book a computer lab weeks in advance (6) - Computers are not used at all (1) - There isn't any software that has been developed for the curriculum (4)

Question 6	Are ordinary calculators used as part of the mathematics class, and if so, how are they used?
	- Most students have them and depend on them in the class (7) - We encourage their use and they are emphasised as part of the curriculum (7)
Question 7	Are graphic calculators used as part of the mathematics class, and if so, how are they used?
	- Graphic calculators are occasionally used by teachers (4) - I encourage the few that have them to use them in class (5) - Graphic calculators are used as part of the class on a daily basis (1) - Most students who have them use them like ordinary calculators, so they don't know how to use them properly (2)

The comments by the teachers in Table 2.1 are combined with classroom observations by the author to discuss and assess each of the factors in Figure 2.1 that could impact on undergraduate mathematics.

2.6 Impact factors

Section 2.6 discusses the various impacts on students going from school to university.

2.6.1 Curriculum and assessment

For most of the observed Year 13 classes teachers gave first priority to previous Bursary examination papers and then to the Bursary examination prescription. The teachers acknowledged that they gave little emphasis to the official New Zealand Mathematics Curriculum, particularly if the curriculum differed from the Bursary examination prescription. This was especially so if the majority of their students intended to sit the 'Bursary Mathematics with Calculus' examination. The number of Scholarships and Bursaries obtained by each school in the Bursary examination is considered to be a yard-stick for a school's academic excellence in the community. Therefore there is pressure for senior mathematics teachers to teach to the Bursary examination. In addition, teachers who obtained a large proportion of Scholarships or 'A' Bursaries from their students felt highly valued.

> *I was given a top class at [name of school] and got so many Scholarships that they (the school) were so pleased they asked me to continue taking the Scholarship class.*

This priority changed to the official New Zealand Curriculum in one class where many of the students did not intend to sit the Bursary 'Mathematics with Calculus' examination. During the interview, the class teacher commented favourably on the flexibility and freedom in being able to diverge from the Bursary Prescription.

Impact: The background content that students bring to university mathematics tends to be dictated in Year 13 by the Bursary Prescription rather than the New Zealand Mathematics Curriculum. Any changes in the Bursary prescription would have an impact on undergraduate mathematics, especially in terms of the background knowledge. A further implication is that the major changes outlined in the New Zealand Curriculum document, such as problem solving using open-ended real life problems, are not being widely implemented.

2.6.2 Teaching and learning preferences

For all except two Year 12 classes in different schools, the classroom was set traditionally with the teacher at the front by the blackboard (or whiteboard) and student desks in pairs or triples facing the front. In the two exceptions, student desks were in groups of four and the teacher's desk was in a section of the classroom away from the board.

Teachers spent from 30% to 60% of their time (for Year 12) and 50% to 90% of their time (for Year 13) in teacher/student discussion. This typically involved class build up of ideas on the board, either by asking general questions of the class or targeting specific students. Exceptions included a Year 13 teacher who spent at least 90% of class time with her students actively doing problem-based exercises and two Year 12 teachers who used groups to work on problem sheets. Teacher/student discussion for most classes was interspersed with five to ten minutes of students writing in their exercise books and working on exercises either from the board or from the textbook. For most classes students were comfortable in seeking clarification from the teacher or questioning the teacher in front of the class.

Apart from the two Year 12 classes that physically moved desks into groups, there appeared to be very little group discussion of mathematical concepts. Most of the students in the more traditional classroom worked individually on exercises and occasionally checked their answers with a nearby student. It was not unusual for any 'group discussions' with fellow students in the immediate vicinity to be unrelated to mathematics or any other academic subject. Students in these classes displayed an initial strategy of asking the teacher for help. Exceptions to this strategy were the Year 13 and two Year 12 classes mentioned in the previous paragraph who were recorded as having low student/teacher discussion time and high written problem solving time. Students in these classes spent much of their class time discussing the work in groups. As a result the students displayed an initial preference to ask fellow students for help and only asked the teacher for help as a final strategy. For each of these classes the average teacher/student ratio was 1:25. The traditional 'lecture' situation of a teacher illustrating concepts without feedback from the class was seen in the statistics class and occasionally seen in Year 13 'top stream' classes (10% to 20% of class time in three Year 13 classes). These classes had ratios of one teacher to 30 students.

Within the classroom teachers used a combination of numeric, graphical and algebraic approaches to varying degrees, depending on the style of the teacher. Most teachers preferred to illustrate concepts pictorially and students being required to complete tables sometimes supplemented this. Teachers also used graphs to illustrate ideas not obvious to the students. For example, in one Year 13 class students were asked to solve for x if $3 < 5\text{-}2x < 11$. The teacher did the following calculation steps on the board:

$$\begin{aligned} & 3 - 5 < -2x < 11 - 5 \\ \Rightarrow\quad & -2 < -2x < 6 \qquad\qquad (1) \\ \Rightarrow\quad & 1 > x > -3. \end{aligned}$$

However, when the students were asked to solve for x when $x^2 + 8x - 20 \geq 0$ the teacher illustrated the solution with a diagram for $(x+10)\ (x\text{-}2) \geq 0$ and emphasised the region for $x < 10$ and $x > 2$.

In most classes students were encouraged to set up tables and approximate answers if they were in any doubt on how to answer a question. Although students perceived the 'right way' to solve a problem was algebraically, they often reverted to a numerical or graphical approach. This was illustrated when a Year 12 teacher gave her students a problem solving exercise. The exercise involved matching equations with questions on related topics such as perpendicular lines, parallel lines, isosceles and right-angled triangles. The students initially tried an algebraic approach in that they attempted to connect ideas with the algebraic equations. However, when they found their algebra was not strong enough to answer more than one part of the problem, they resorted to graphing all the equations. In doing this, most of the students set up tables and plotted points. One group achieved the overall solution and this success provided the competitive edge for the other groups.

Teachers, especially Year 13 teachers discussed some mathematical theory, but this was usually limited to definitions. There was a marked absence of formal or informal explanations of concepts even for the 'top stream' classes. Teachers knew that they could safely avoid the limited number of basic proofs required in the Bursary examination prescription.

<u>*Impact*</u>: There would be a dramatic transition from small group teacher/student situations to a large lecture environment. Unlike the schools with average teacher/student ratios of 1:25, the university lecture environment has class ratios of 1:250. The teaching styles differ and students need to adjust to significant changes. Student preferences can also influence the transition to university mathematics. The school students preferred a numerical and graphical approach. University mathematics usually takes an algebraic and theoretical approach. The difference in teaching/learning approaches between school and university can explain some of the current difficulty students encounter in the transition to university mathematics. Absence of, or limited access to, theory and proof in schools could also explain the negative reaction of students to theory in first year university mathematics.

2.6.3 Resources

Common resources available to students were textbooks, and technology such as traditional calculators, graphic calculators and computers. The new mathematics curriculum emphasises the use of new technology.

(a) Textbooks

Overall, the textbook was used for routine exercises both in class time and for homework. All schools used the same text and because the text sparingly explained many of the mathematical concepts, only one teacher (in a 'top stream' Year 13 class) used the textbook for students to study concepts. This teacher briefly highlighted any theoretical aspects using the text and then expected the students to study the text further for deeper understanding. All other teachers developed their own notes on the board and did not refer to any theoretical parts of the text.

Impact: Use of textbooks solely for routine exercises may explain the reluctance of most students to improve their conceptual understanding by using this resource in first year mathematics. If any students were encouraged to use textbooks for understanding they tended to be in the 'top stream' category.

(b) Calculators

Students had become totally dependent on calculators and the teachers encouraged frequent calculator use. In each classroom 85% to 90% of the students had calculators on their desks and intermittently worked them about 75% of the time during class. Although those students without a calculator on their desks had purchased their own instrument, the students often relied on their neighbour to do the calculations for them both. In six classrooms there were at least five students who held a calculator in one hand and a pen in the other. While the teacher was developing ideas on the board at least three students in every classroom were checking the teacher's calculations without being asked to do so. It was not uncommon for teachers to ask for the answer from those students holding a calculator.

Students used the calculator for even the simplest calculations including basic multiplication, addition, subtraction and division. For example, $\frac{0}{1}$ or -3 minus 2 . When a teacher did an example on the board that required simplification such as $\frac{\log 64}{\log 4}=\frac{\log 4^3}{\log 4}=\frac{3\log 4}{\log 4}=3$, four students out of 25 students checked the answer on their calculator rather than follow the algebra.

Students also tended to use their calculators once and seemed averse to making the most of calculator speed by rechecking answers. For example, one student used his calculator and obtained an incorrect answer. He did not recognise this at the time and waited for the teacher to give the answer to the class. When the student found his answer was incorrect, he wrote the correct answer beside his work but did not attempt to recalculate or discover where he made the error. Other students checked their answers with each other rather than rechecking their calculations on the calculator.

<u>*Impact:*</u> The dependence on calculators in senior mathematics for even the simplest calculations can have a large impact on undergraduate mathematics. Several of our first year mathematics classes at university are not permitted to use calculators in tests or examinations. Student stress as a result of this ruling can be explained by the school observations. Another impact comes from comments by students that confirmed the belief that the calculator was always correct and that the calculator gave answers for everything they needed to know.

(c) Graphic calculators

In every class there was at least one student who owned a graphic calculator. Although many of the teachers said they encouraged their use, they also commented that many of the students who possessed a graphic calculator tended to use it like an ordinary calculator. In one class where three students possessed graphics calculators, only one student used it to check on a graph she developed by using a table. In one school graphic calculators were borrowed once or twice a year from a Government funded unit (Education Resources). In another school a set of graphic calculators, including one for overhead projector (OHP) demonstrations, had been lent to the school for an indefinite period. In this latter school, the students were taught how to use the calculators as part of their course, and the teacher used the OHP demonstration calculator for most of the mathematics classes. This situation, however, was not common.

<u>*Impact:*</u> Very few students possessed a graphic calculator and many students commented on cost as an influencing factor. Since many of those that did purchase a graphic calculator did not know how to use one to its full potential, the use of graphic calculators would not result in any immediate impact on university mathematics. However, if more schools had access to class sets, as did one of the schools observed, if graphic calculators became cheaper and became an integral part of the Year 13 curriculum, then the universities would have to seriously consider incorporating the use of graphic calculators in the implementation of their syllabus.

(d) Computers

In all the schools observed, computers were not used in senior mathematics classes, especially the Year 13 classes. Some of the Year 12 teachers used the computer for specific tasks. Most teachers commented on the inconvenience of the computers and lack of software associated with the new 1992 curriculum. Teacher comments included:

> *We have two computer labs, but we need to book about three weeks in advance. By the time we get to the lab, we are doing an entirely different topic in maths.*

> *We don't really have any good software to use with the curriculum. We have spreadsheets, but that is all.*

Two teachers commented that they would only consider using the computers for selected topics such as functions, conics and perhaps numerical integration. However, since computer use involved students moving out of the mathematics classroom, a special computer laboratory lesson had to be specifically designed for a 50 minute period. Many of the teachers felt this was not worth the effort. This attitude once more reinforced that the teaching is being driven by computer-free Bursary prescriptions.

Impact: There is no immediate impact on undergraduate mathematics from computer use, but the potential exists if software is developed for the secondary curriculum and if computers were installed in each classroom or easily available to each student. Although many secondary students are proficient in word processing and to a lesser extent in spreadsheet software, at the senior level these are not used as yet as an integral part of the mathematics classroom.

2.6.4 Student skills: problem solving and algebraic skills

The Ministry of Education's mathematics curriculum emphasised the mathematical process skills of problem solving, reasoning and communicating mathematical ideas (Ministry of Education, 1992). These processes were emphasised in the curriculum's five strands of content labelled "number, measurement, geometry, algebra and statistics".

(a) Problem solving

For most teachers, especially those involved in teaching Year 13 students, 'problem solving' was interpreted as solving problems expressed in 'word form' predominantly for the differential applications section of the curriculum. Two Year 13 teachers emphasised the general process of problem solving within everyday mathematical exercises. One of these teachers had a large poster with the basic problem solving steps displayed on the classroom wall facing the students. The teacher used this poster four times in one lesson to encourage the students to consider the 'next step' in solving a problem. The other Year 13 teacher kept referring to critical problem solving steps as he did a homework question on the board.

The author noted that for many of the problem solving exercises given in Year 12 classes, the teacher assumed the students were already familiar with the basic problem solving steps. The students however did not appear to have any of the problem-solving heuristics described by Fogler and LeBlanc (1995) as a 'systematic approach that helps guide us through the solution process and generate alternative solution pathways' (p. 8). The main strategy used in the classrooms appeared to be learning by discovery, usually resulting in the student taking longer to solve a problem.

Impact: 'Problem solving' using real-life problems is still not emphasised in schools, despite recommendations in the Curriculum document. Some problem solving skills were being developed in the secondary sector but this varied with the teacher. As a result of the avoidance of real-life problem solving any impact on undergraduate mathematics is minimal.

(b) Algebraic skills

Overall, the algebraic skills of students were not strong, despite a month at the beginning of Year 13 devoted to a revision of algebra that usually finished with a test. It was not uncommon for Year 13 teachers to skip over algebraic steps while doing examples on the board. These teachers often dismissed any algebra as "work you (the class) should already know". Although algebra was emphasised in Year 12 classes, only one Year 13 teacher emphasised the importance of algebra both in instruction and in student work. All but one of the teachers, however, mentioned a concern with student algebraic skills generally.

On one occasion a teacher gave the students some exercises to do in class. The author observed one student with the incorrect answer. The student could not see why his answer was incorrect. The teacher answered the student's request for help by replying:

> *Oh, you have probably made a minor algebraic error.*

In fact, the student had replaced $\frac{3}{2+\sqrt{2}}$ with $\frac{3}{2}+\frac{3}{\sqrt{2}}$. The student also had written $3\sqrt{2}+\sqrt{50}=\sqrt{58}$. The reasoning given was that $3\sqrt{2}$ became $\sqrt{2^3}$. So $\sqrt{2^3}+\sqrt{50}=\sqrt{8}+\sqrt{50}=\sqrt{58}$.

Therefore a further algebraic error came with the addition of square roots. This comment from the teacher reflected the likelihood of algebraic manipulation not costing too many lost marks in end-of-year external examinations.

In another class (Year 12) a student asked the teacher if $\frac{2}{x^3}=\frac{2^{-3}}{x}$ was correct. The reply came immediately from another student stating that the equation was correct and this view was supported vocally by other students in the class. Even when the teacher explained why the equation was incorrect, there was an obvious feeling of disbelief from the class as a whole. Many students in this class were observed having difficulty recognising what to do, applying appropriate rules and manipulating algebraic expressions.

Impact: The assumption that all students had already mastered years of algebraic skills, the assumption that incorrect solutions were due to 'minor algebraic errors' and the over-dependence on calculators, can all have a potentially negative impact on the algebraic skills that students bring to university mathematics.

2.7 Summary

Observations and interviews in the schools explained some of the current difficulties in the transition from secondary to tertiary mathematics. Examples include student reluctance to use textbooks for anything other than routine exercises; the reluctance of students to use resources other than the lecture notes to develop understanding of concepts; the dependence on calculators and the confusion when calculators could not cope with more complicated functions; some weaknesses in algebraic skills and the unfamiliarity of simple proofs and theory.

Of these, the ones that seemed the most significant to the author were algebraic competence, calculator dependence for even the simplest arithmetic calculation and a lack of textbook use to read and comprehend mathematics. This impression came from a combination of the school observations outlined in this chapter and ten years tutoring and lecturing experience with undergraduate mathematics students. The potential impacts include computers, provided software and accessibility are improved and graphic calculators when the cost of these becomes affordable.

Chapter 3 Undergraduate Algebraic Skills

3.1 Introduction

Much of the research on algebraic skills and their classification has focussed on analysing specific types of errors or misconceptions, predominantly at the primary or secondary school level. A large body of research occurred in the 1970's and 1980's, for example, Ashlock, 1976; Becker, 1988; Cox, 1975; De Corte & Verschaffel, 1981; Engelhardt, 1977; Kilian, 1980; Radatz, 1979; Resnick, 1984; Sleeman, 1984; Sleeman, 1986; Stefanich & Rokusek, 1992. While some researchers focused on classifying the gaps students had in mathematics learning, others concentrated on the cognitive processes that underlie the errors.

Research in gaps in mathematical knowledge involved the study of errors, focused either on one aspect of mathematical content such as limits, or on basic computations (e.g. addition, subtraction and rational numbers), or on the introduction of students to variables. Much of the research included classification of errors and the study of simple word problems. The basic idea that emerged from literature was that errors were not only systematic, but could be eliminated by remediation.

3.1.1 Error classification

Until the mid 1980's most studies on error classifications were conducted at the basic computational level, namely up to three digit addition, subtraction, division, multiplication and rational numbers. As early as the 1920's Bruechner did extensive work on identification of types of errors in computation (cited in Carr, 1986). In his research Bruechner tabulated 8785 errors into 14 categories, many of which were later found to be ambiguous (Ashlock, 1976). Other researchers have also classified errors and these included Babbitt, 1990; Bell, 1982; Cox, 1975; De Corte & Verschaffel, 1981; Engelhardt, 1977; Kilian, 1980; Pinchback, 1991; Sleeman, 1984; Stefanich & Rokusek, 1992. The classification that appeared to have dominated the literature was cited in Cox (1975) whose work was an expansion of Grossnickle's (1935) research. Cox divided Grossnickle's two categories of 'constant errors' and 'errors due to chance' into three categories 'systematic', 'random' and 'careless'. Systematic errors in computational mathematics have also been known as algorithm errors, procedural errors or 'bugs'. The jargon for careless errors became known as 'slips'. Much of the research at the time assumed that 'bugs' reflected systematic stable mistaken beliefs about a skill, while 'slips' were non-systematic performance phenomena, unstable over time and loosely related to the problems in which they occurred (Van Lehn, 1982). Similar classifications were used by researchers such as Stefanich (1992) who found that of the 25 ten year olds studied, 47% displayed careless errors and 38% systematic errors. However, even careless errors could be systematic. For example, Kilian (1980) found that in the analysis of 121 primary students' scripts, errors that appeared to be random or careless displayed a systematic pattern. Likewise, Lankford in his interview-based study on whole numbers and fractions with seventh graders found mainly systematic errors, although it was noted that these errors frequently accompanied unauthorised strategies (Lankford, 1974). However, even at the time not all researchers acknowledged the stability of errors. For example, Van Lehn used a computer diagnostic program (BUGGY) and found that one-third of the errors committed by students could not be modeled by the program (Van Lehn, 1982). Van Lehn concluded that many 'bugs' were unstable.

The question is whether any or all of the classification research in lower level mathematics can be extended to higher level mathematics. Most research on error classification basically assumed that students reason upwards from findings in elementary work. This is reinforced by research into the difficulties involved in the transition from arithmetic to algebra, for example, (Booth, 1988; Filloy & Rojano, 1989; Gagatsis & Christou, 1997; Harper, 1980; Herscovics & Linchevski, 1994; Pirie & Schwarzenberger, 1988). However, pockets of recent research maintain that higher level mathematics is so different that elementary arithmetic research cannot explain the difficulties in higher level mathematics (Rotman, 1991).

Some research on error classification focused on higher level mathematics rather than lower level arithmetic. For example, in the work by Radatz and an extension of this work by Pinchback (Pinchback, 1991; Radatz, 1979), Radatz presented five types of errors that appeared to have the potential to extend beyond basic computation. These included errors due to language difficulties, errors due to difficulties in spatial information, errors due to deficient mastery of prerequisite skills and facts, errors due to incorrect association or rigidity of thinking, and errors due to irrelevant strategies. Likewise, Pinchback analysed errors made by her remedial algebra class of college freshmen and broadly divided the errors into 'conceptual' and 'prerequisite'. Conceptual errors occurred when the student attempted to apply correct procedures but made errors in the procedure itself. Prerequisite errors displayed deficiency in knowledge of prior concepts and combined two of Radatz's categories: that is, errors due to rigidity of thinking and errors in application of irrelevant strategies. In Pinchback's research just over half the errors in the study were 'conceptual' rather than 'prerequisite'.

Other research in higher level mathematics concentrated on the classification of errors that could arise from problem solving strategies, especially the transferring from a word problem into algebraic notation. Two similar studies were done by Rosnick (1981) and Clement (1982) who found that of the 150 engineering students (plus 45 non-science university students in Clement's study) relatively advanced students could experience serious difficulties in symbolising meaningful relationships with algebra questions. They found that 73% and 68% of the students reversed the problem in translation to numeric form, e.g. 6S=P instead of 6P=S, and therefore labeled this type of error as 'reversal errors'. Pinchback (1991) also mentioned this type of error.

About the time error classification was at a peak, there was an increasing focus on cognitive research, in particular how students understood mathematics, how misconceptions could be formed from interference and how errors could be formed from misconceptions. For example, Resnick and Matz were two of many researchers who assumed that the underlying problem of all student errors lay in the way the student understood or misunderstood mathematical concepts (Matz, 1980; Resnick, 1984). Errors were seen as the reflection of student misconceptions and the way to study these errors involved taking a cognitive approach to explore how students learn to comprehend mathematical concepts. Matz was one of several researchers who developed a theory of mathematical competence and proposed that errors were a result of reasonable, although unsuccessful attempts to adapt previously acquired knowledge to a new situation. This theory was developed to explain the common systematic errors that seem to reflect either unmade developmental changes or an incorrect choice of an (otherwise correct) extrapolation technique. On confronting a new situation, students either adapted an old rule or saw a new problem as an old familiar one. Such adaptations had the potential to lead to a diversity of incorrect and correct answers.

Since the mid-1990's increasing focus has turned specifically towards algebraic difficulties in tertiary mathematics, for example, (Barbeau, 1995; Edwards, 1995; Kaur & Sharon, 1994; Pinchback, 1991). Research has shown that students entering university courses with a substantial mathematical background not only have difficulties with algebra but may have fewer algebraic skills than was the case a few years ago (Hunt, 1996). There is also concern that more and more students lack the ability to conceptualise tertiary algebra in the face of increasing evidence that skills could be secured only when some understanding is in place (Barbeau, 1995). Attempts to explain algebraic thinking cognitively through to the tertiary level have been helped by various theories put forward by researchers such as Matz, 1980; Sfard, 1991; Sfard, 1995; Sfard & Linchevski, 1994; Tall, Thomas, Davis, Gray, & Simpson, 1998; and Dubinsky, 1991.

3.1.2 Theories

The theorists mentioned above take a constructivist perspective with an underlying assumption that students actively construct knowledge internally and that such knowledge requires further reconstruction through a multi-linking of pre-requisite and incoming knowledge. At the university level, the new knowledge that is presented to students requires time for assimilation outside formal contact hours. Therefore much of the students' mathematical knowledge and understanding of concepts depends on how they construct and integrate their knowledge with the information presented to them. Many of the theories appropriate to this study explain algebraic processing by extending and reorganising aspects of Piaget's work. Their ideas involve active construction of mental processes and the encapsulation (or reification, (Sfard, 1991; Sfard, 1995; Sfard & Linchevski, 1994)) of mental processes into static structures. Obstacles to this encapsulation process are put forward as explanations of algebraic errors, misconceptions and difficulties (Thomas 1994).

3.1.3 The purpose of the algebraic skill study

At our university in New Zealand, we are also facing similar difficulties with poor algebraic skills as an inheritance from our schools. Interest for the author initially arose from marking mainstream university mathematics first year papers. When students who come from diverse backgrounds consistently make similar types of algebraic difficulties, the argument for examination 'slips' becomes hard to accept. This argument is even less convincing when the difficulties re-emerge with each yearly intake of first year and second year students. More research was needed on defining what exactly were the widespread consistent difficulties that we faced at the tertiary level. Did higher mathematical learning affect these algebraic skill difficulties and is there a possible framework for remediation at the tertiary level?

The purpose of this chapter, therefore, is to define the most common widespread difficulties first year university students in New Zealand experience with algebraic skills. The analysis in this study extends ideas by (Rotman, 1991) who looked at algebra pre-requisites. The next chapter (Chapter 4) explores the effect of higher learning on the algebraic difficulties.

3.2 A broad approach to algebraic skills

Algebra is not just algebraic manipulation and applications of rules, it involves looking at *why* as well as *how to do* algebra (Kaur & Sharon, 1994). The Collins Mathematical Dictionary (Borowski, 1989) defines algebra as a generalisation of arithmetic using variables. From another perspective, the 'Mathematics in the New Zealand Curriculum' document (Ministry of Education, 1992) says that algebra should enable a recognition of patterns and relationships especially in real-world problem solving. However, real world problem solving is pointless if the student cannot comprehend or manipulate the expression correctly in order to obtain meaningful real world solutions. Therefore a definition needs to include not so much the generalisation from arithmetic or the algebraic expression of real world problems, but the interpretation and understanding of algebraic symbols, expressions and solutions. For example, a lack of conceptual understanding may occur when students consider *cos x* as two separate entities, *cos* and *x*. This is just one example of a prerequisite error (Pinchback, 1991), that is, conceptual errors that occur when previous concepts have been insufficiently mastered. More broadly, Sfard (1995) describes algebra as a science of generalised computational processes and concepts where one progresses to deeper levels of insight and sophistication. Such an approach is supported by (Barbeau, 1995; Tenzer, 1983). The definition of algebraic skills in this study therefore includes levels of understanding of processes and procedures and an extension of skills into more abstract concepts.

3.3 Placement tests and algebra prerequisites

Although the value of placement tests is not the focus of this study, analyses of placement tests have contributed considerably to the literature on algebraic difficulties. Increasing concern about a lack of algebraic skills has led to prerequisite assessment for entry to university. This is predominantly a response to the increased enrolment into undergraduate mathematics of students possessing a wide variety of mathematical backgrounds. In some universities and colleges, students sit placement tests that determine whether they should be first placed in courses designed to bridge gaps in basic arithmetic or algebraic skills. The tests may be multi-choice with specific distracters (Edwards, 1995) or a combination of multi-choice, true/false, mistake detecting or short answer items (Kaur & Sharon, 1994).

There is some evidence to suggest that the placement of students into predominantly *arithmetic* bridging courses may not significantly improve student performance later in mainstream algebra courses (Rotman, 1991). Rotman's work came from research and observations at Lansing Community College (Michigan). He compared test scores and success rates between an arithmetic pre-algebra course, a beginning algebra course and an arithmetic placement test. The courses used self-study plus a competency-based testing system. Rotman found that the arithmetic test scores did not strongly correlate with student performance in the beginning algebra course and concluded that, for a large body of students, there was not any advantage in completing an arithmetic course as an algebra pre-requisite. His work reinforced evidence by Lee and Wheeler (1989) on the dissociation between arithmetic and algebra at the secondary level. However, Rotman noted that there appeared to be *some* connection between *certain* arithmetic skills and performance in algebra. He proposed the following four basic arithmetic skills as algebra prerequisites.

- Understanding the meaning of symbols used in arithmetic.
- Understanding the basic properties of numbers, especially fractions.
- Using the order of operations agreement.
- Understanding some of the structure behind solving equations.

Rotman's categories were proposals resulting from observations rather than from an analysis of algebraic skills. For the category, *understanding the meaning of symbols used in arithmetic,* Rotman maintained that we need to know more than how to manipulate, especially as arithmetic knowledge may differ from algebraic knowledge. For example, 23 represents the addition of $20 + 3$ in arithmetic, but $2a$ represents a product in algebra.

Rotman stated that there was a need to examine the common difficulties that students encountered with algebra with "an eye toward what effect arithmetic instruction has had on them" (Rotman, 1991, p.12). This study examines common difficulties with a focus on discovering to what extent Rotman's pre-algebra categories existed at the tertiary level.

3.4 Method

At our university in New Zealand, we use the final secondary year (Year 13) calculus examination as a placement test for first year undergraduate mathematics. First year mathematics students are placed into three classes based on their marks (that is, either <50%, 50%→75% or ≥75%). Basic algebraic skills are considered a significant problem even for those students who gained more than 50%. The students who took part in this 1996 study were from the 50%→75% and ≥75% classes and the author therefore expected them to have some algebraic competence already established. The students sat an algebraic test at the beginning of the year. Later in the year, the author supplemented this work by inserting questions into informal help sessions. There were more than 50 students who contributed to the list in the section labelled Section 3.7 at the end of this chapter.

3.4.1 The groups

Short answer, 36-item pen and paper algebraic tests were administered to each of two groups of first year university students, labelled for the convenience of this study as Groups A and B. The two groups were divided according to their marks gained from the final year secondary school calculus examination. Group A students obtained more than 75% in the examination and Group B students gained between 50% and 75% in the same examination. There were 88 students in Group A and 526 students in Group B. Most of the students were 17 to 18 years of age and, at the time of this study, both groups were currently enrolled in full year mainstream university mathematics courses. The algebraic tests covered some of the routine competency skills assumed to be mastered prior to entering university. The tests were therefore based on algebraic skills that the lecturers' believed were taught in the secondary schools (Appendix A). They were administered in the third week of the academic year, after a four month summer recess between secondary school and entry to university.

3.4.2 Test conditions

The test conditions differed slightly between the two groups. Group A students sat their test in normal lecture time (50 minutes) and were given 25 short answer questions, that is two minutes per question. Group B students sat their test on a weekend. This group was given 90 minutes to complete a 36-item short answer test. The lower scoring group (B) was therefore given slightly longer to complete each question. Test marks for Group B contributed 5% towards the final course grade while the test results for Group A did not contribute to the final grade. Calculators were not permitted in the tests and students were given a table of formulae instead.

3.5 Analysis of tests

The analysis of the 1996 tests concentrated on a variety of topics such as trigonometry, limits, differentiation and integration. Responses for each test item were initially placed into five categories labeled: *correct response, did not answer, did not complete but correct so far, unknown* (where the solution could not be obviously related to the question), and *incorrect response*. The last category contained between five and 12 subgroups of different errors depending on the item.

Group B students consistently displayed more subgroups of incorrect responses than the more able Group A students.

3.5.1 Topic comparisons

Test items were collated into 13 major topics and the average proportion of correct scores was calculated per topic (Figures 3.1 and 3.2).

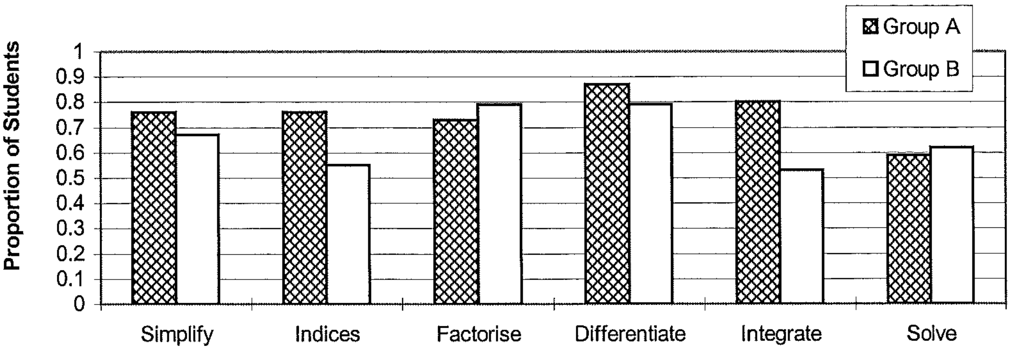


Figure 3.1: Average proportion of correct responses for six of the topics; Group A versus Group B students

As expected, Group A students displayed a higher proportion of correct responses in most of the topics than Group B students (Figure 3.1). There appeared to be exceptions with factorising and solving equations. However, for each of these topics, the question for Group A students was more difficult than the equivalent question for Group B students. For example, Group A students were asked to factorise the difference between two cubes. Only 24% of Group A students gave the correct response, 27% did not give an answer and 25% displayed a variety of solutions, predominantly with $(x^2 - y^2)$ as a factor. Group B students were asked to factorise the difference of two squares. Likewise, in solving for *x*, Group A students were asked to solve two simultaneous equations $3x^2 + y^2 = 1$ and $2xy = 0$. Only 27% of the Group A students gave the correct response and 28% omitted $\pm$ in their answer. If a similar question is asked of Group B students the proportion not answering the question is likely to be much higher.

In Figure 3.2, trigonometry was the topic causing most difficulty for both Groups even though the question for each group should have been straightforward for their level. That is, if $\sin x = \frac{1}{2}$ what is the value for $\cos x$? This type of question required students to understand the basic sine and cosine functions, the standard $(1,2,\sqrt{3})$ triangle and/or some of the trigonometric identities, namely $\cos^2 x + \sin^2 x = 1$. The most common incorrect response from both groups was the assumption that the question referred only to the positive value of x where $0 \leq x \leq \frac{\pi}{2}$.

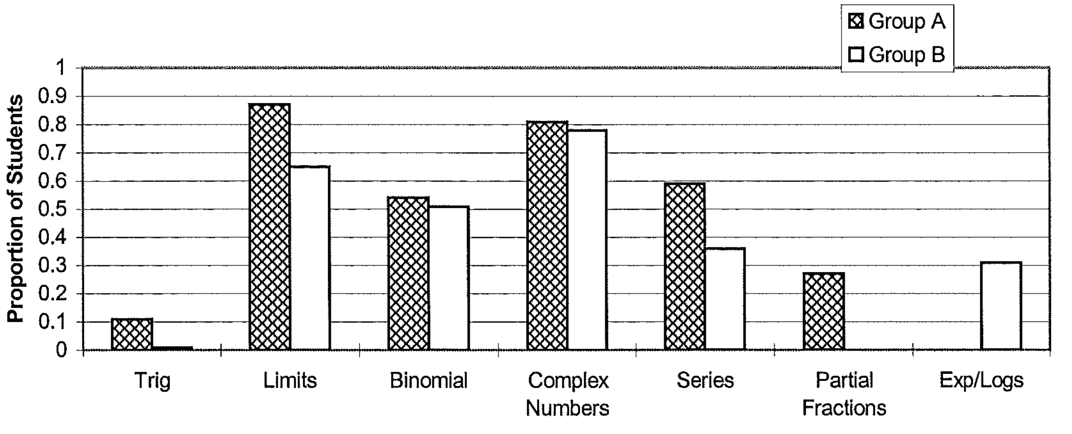


***Figure 3.2:** Average proportion of correct responses for seven of the topics; Group A versus Group B students*

Only 0.6% of the students in Group B and 11% in Group A obtained the correct answer. A further 53% in Group A and 40% in Group B did not consider the negative value for cos x. This type of incorrect response was not solely limited to trigonometry, it seemed to be topic independent and widespread enough to become the basis for one of the five categories.

The topics in Figure 3.2 emphasise the mismatch between the tertiary lecturers' expectations and student prior knowledge. The limits and series questions were handled better by the more able Group A students than Group B students (Figure 3.2). Both these topics were expected to be briefly encountered at a rudimentary level in the final secondary school year and are considered by all students to be among the most difficult topics to understand in first year university mathematics. These results therefore show that the same difficulty experienced at the end of first year mathematics by Group B students was already a problem in senior secondary mathematics.

One question on partial fractions was given only to Group A students. Partial fractions is no longer included in the Bursary syllabus and is therefore likely to be taught only to an accelerated class of secondary students. Questions on exponential and logarithmic functions were given to Group B alone as this topic had consistently caused difficulty with first year students in previous years. The analysis confirmed that these topics had not been mastered prior to university entry.

3.5.2 Topic independent comparison

It soon became apparent that similar difficulties existed across all the topics tested. These difficulties were collated, culminating in a regrouping and reassessment of categories. The categories were established for responses that were in common with at least 10% of the students. The five categories that emerged are in Table 3.1 and most of the categories are similar to difficulties that have already been mentioned by researchers. For example, the first three categories are similar to those found in the list proposed by Rotman (1991). This study therefore confirmed most of Rotman's categories and implied that some difficulties appear to be directly linked to understanding some essential pre-algebra arithmetic skills.

Table 3.1: Categories of algebraic skill difficulties

Category	Label
1	Ability to apply the order of operations agreement, especially the role of brackets.
2	Ability to apply the properties of numbers, especially fractions and rational functions.
3	Ability to follow the structure of the underlying procedures.
4	False generalisation.
5	Judgement in exploring the range of possible solutions.

The first category in our study, *ability to apply the order of operations agreement* is also mentioned by Kieran (1979), Booth (1988) and Davis (1984), where the researchers found that children strongly believed the written sequence of operations determined the order for performing the calculation. The third category in the study, *the ability to follow the structure of the underlying procedures* is also mentioned by (Küchemann, 1981) who looked at the difficulties students experienced in interpreting algebraic notation and manipulating variables. The same symbols can take different meanings, depending on the context in which the symbols are presented. This is also illustrated in the work by Teppo & Esty (1995) who looked at the four different types of meanings the quadratic theorem could take when the theorem is presented in four different contexts. Category 4 in this study, *false generalisation*, is again mentioned in work by Davis (1984) who found that if there was little understanding of concepts learned in one situation, the techniques or tools the students use in a different situation may be inappropriate. He termed *false generalisation* as a Frame-Retrieval Error in that retrieval of knowledge learned in one situation is inappropriate for the current task. Category 5 in this study, *judgement in exploring the range of possible solutions*, does not appear to be associated with typical algebraic skills and implies the use of judgement skills in interpreting solutions.

3.6 The five categories

This section gives a rationale for the five categories proposed in Table 3.1. Interpretations are seen from the undergraduate student point of view. Although researchers have previously mentioned these categories, they have done so from a primary or intermediate student perspective. It is the contention in this study that undergraduate students not only have considerably more mathematical knowledge but they should be more aware of the process of their algebraic difficulties.

3.6.1 Category 1: Ability to apply the order of operations agreement, especially the role of brackets

At the primary and early secondary level students learn the order of operations agreement. In our tests, treating brackets as if they did not exist and not using brackets to group variables, stood out as a consistent cause of difficulty. Students also failed to realise that brackets used in arithmetic could play a different role from those used in algebra. In this way, Category 1 describes an incorrect view of an arithmetic representation in algebra (Booth, 1988) or dissociation between arithmetic and algebra (Collins, 1989). For example, in arithmetic, a bracket indicates an order of operation as in $1-(1+\frac{1}{2})$ where the operation $1+\frac{1}{2}$ takes priority over subtraction. In algebra, although the same order of operation still applies, the operation within a bracket may already be simplified as far as possible, for example

$$1-(1+\frac{x}{2}-\frac{x^2}{3})=0 \qquad (1)$$

This time, 'removing brackets' amounts to using the distributive property to get

$$1-1-\frac{x}{2}+\frac{x^2}{3}=0 \qquad (2)$$

A common error found in this study occurred when students ignored the brackets and consequently did not use the distributive property, for example,

$$1-1+\frac{x}{2}-\frac{x^2}{3}=0 \qquad (3)$$

Since calculation taking first priority (within brackets of equation (1)) could not be evaluated, students employed their second priority, subtraction. This difficulty appears to be more than just visual cues not being sharply distinguished (Davis, 1984) but rather a deliberate decision to use what is considered to be an appropriate alternative strategy to resolve a conflict. This conflict comes from a lack of simplification within brackets and from the -1 in equation (1) being implied rather than explicit. This weakness could relate to Davis' *pre-mature tree-pruning error* where the student discards or overlooks an important possibility.

A further example was seen when students were asked to differentiate $y=e^{(-x^2+2x)}$ and their answer is $y'=e^{(-x^2+2x)}-2x+2$. Only for some students is the bracket around (-2 x + 2) implied. (See student comments in Section 3.7)

Students who rote learn the order of operations agreement and blindly apply the same strategy to algebra display a surface learning approach. For this category there is a need to determine appropriate strategies and acknowledge that the operations in arithmetic may be different not only in algebra but also in different algebraic situations. This implies the necessity for a deeper strategic approach.

3.6.2 Category 2: Ability to apply the properties of numbers, especially fractions and rational functions.

Fractions are encountered initially in the primary school. The processes learned at this level can impact later on algebraic skills, especially since much of first year university mathematics involves operations on rational functions. Of the five categories this is the one that most closely resembles a direct transfer of knowledge and procedural manipulation from arithmetic to algebra.

In this category there were two main sub-levels: simplification of rational functions and addition (or subtraction) of rational functions. For example, when asked to rearrange

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2} \text{ into } R_1 = \ldots. \qquad (4)$$

one third of Year 1 university students inverted the fractions separately before rearranging. That is,

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2} \Rightarrow R = R_1 + R_2 \qquad (5)$$

This answer appears to represent an inability to comprehend reciprocation of fractions, a process in arithmetic that can be extended to algebra. Others (16%) left their answer in terms of $1/R_1$. Again, students did not distinguish between a variable and its reciprocal. Still other common answers were

$$R = R_1 - R_2 \text{ or } R = \frac{1}{R_1} - \frac{1}{R_2} \qquad (6)$$

Evidence that this difficulty stems back to arithmetic was also seen in other parts of the tests. For example, in the questions relating to indices most of the students could change from root to index form, that is

$$\sqrt{x}.\sqrt[3]{x} = x^{\frac{1}{2}}.x^{\frac{1}{3}} \qquad (7)$$

However 22% of Group A and 26% of Group B students could not add the indices $\frac{1}{2}+\frac{1}{3}$ correctly. Even further evidence was displayed in other calculations such as $\frac{1}{2+3i}$ or $\frac{1}{2+\sqrt{3}}$ where 8% of Group B and 2% of Group A students wrote $\frac{1}{2}+\frac{1}{3i}$ or $\frac{1}{2}+\frac{1}{\sqrt{3}}$. Some of the Group B students (14%) resubmitted a copy of the question as their answer. It was not clear from the scripts whether this was their genuine answer to the question or whether the student employed a well-used undergraduate strategy of always rewriting the question if in doubt.

Likewise in the simplification of rational functions, another type of incorrect response for 8% - 13% of the students occurred when $2n^2$ was assumed to be the same as $(2n)^2$, indicating a possible connection with Category 1. However, this response was placed in Category 2 because of the subsequent cancellation that followed the Category 1 error. That is:

$$\frac{\cancel{2n+}\,2n^2}{\cancel{2n}} = 2n^2 \quad \text{or} \quad \frac{\cancel{2n}+2n^2}{\cancel{2n}} = 2n\ . \tag{8}$$

Similar types of errors were classified by Becker (1988) as an 'erroneous application of an operator' where the error was said to be caused by neglecting conditions in applying an operator. However, this author sees Category 2 as displaying a fundamental problem not only with the processes involved in rational functions but with the understanding of the arithmetic behind the processes, as suggested by Rotman. On adding rational functions, there was difficulty obtaining the correct denominator, and even if the denominator was correct, some of the students in each cohort could not then calculate the numerator correctly. Surface learning of the arithmetic processes relies on obviously familiar cues in more complicated rational functions. These cues are often not there and in this way competence with processes combined with understanding rational functions should overcome many of the difficulties in this category.

3.6.3 Category 3: Ability to follow the structure of the underlying procedures.

When colleagues refer to 'poor algebraic skills' they often mean that the students make errors in procedures or algebraic manipulation. Included in Category 3 is the control of the use of formulae, manipulation of variables within equations and the deeper understanding of what symbols mean within the context in which they are presented. Although there are many examples that could illustrate Category 3, the study focused on one specific example: the calculation involved in solving a simple quadratic equation. The most common incorrect response occurred in the adjustment of constants. For example:

$$\begin{aligned} & x^2 - 9x + 8 = 0 \\ \Rightarrow \quad & (x^2 - 9x + \frac{81}{4}) - 1 = 0 \\ \Rightarrow \quad & (x - \frac{9}{2})^2 - 1 = 0 \end{aligned} \tag{9}$$

This type of incorrect response was seen in 5% of Group A and 23% of Group B. Students did not comprehend that their adjusted equation needed to be equivalent to their original equation. There were two main errors. One was found in adjusting b in the equation $(x+a)^2+b=0$ (for 12% of students) and the other involved difficulty with using the quadratic formula (for 13% and 18% of students). Another common problem with the quadratic formula was the gradual shrinking of the divisor line (for 8% and 12% of students), or miscalculation of the value under the square root (5% and 6% of students). The shrinking of the divisor line just mentioned occurs when the formula, with or without numerical values substituted for *a, b* and *c,* changes from

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a} \tag{10}$$

$$\text{to } x=-b\pm\frac{\sqrt{b^2-4ac}}{2a} \qquad \text{or} \qquad x=\frac{-b}{2a}\pm\sqrt{b^2-4ac}.$$

Often this is a gradual process over several lines of calculation reflected a surface procedural approach to applying the formula. There is little understanding of the mathematical changes imposed when an equation is carelessly altered.

Although there are numerous obstacles involved in performing these algorithms the main obstacle in Category 3 appeared to be the lack of deeper understanding of what a student can and cannot do in a calculation. Not only do a variety of different arithmetic and algebraic processes and concepts need to be assimilated and understood prior to use, but so does the comprehension of "=" and "⇒". In addition, judgement or control skills play a large role because understanding the underlying idea behind procedures involves the coherent use and combination of different knowledge units. Difficulties in this category may therefore not be easily explained in terms of any one type of classical error but may be explained as a control error (Matz, 1980) where the mechanism for overseeing and controlling an algorithm is defective.

3.6.4 Category 4: False generalisation

This category describes the situation where students who learn concepts in one context can experience difficulty in placing those same concepts into a different context. At university students have been known to select an inappropriate 'tool' or formula and incorrectly alter that formula to fit a question. This type of weakness was described by Kaur & Sharon (1994) as the 'formulation of false generalisations from known laws' and Davis (1984) as 'frame-retrieval errors'. In this situation, a deeper understanding of many concepts and the interrelationships between those concepts is paramount. An appropriate illustration occurred during the observation of senior secondary classes (Chapter 2) where one student wrote

$$\frac{(1+t)^2}{t^2}=\frac{1+t}{t} \tag{11}$$

and described how he had cancelled 'like' indices. The student used one rule and applied it to an inappropriate context. Likewise in the 1996 tests, first year university students were asked to solve both

$$(3x - 4)(x + 1) = 0 \quad \text{and} \quad (3x - 4)(x + 1) = 2. \qquad (12)$$

For the first question, 93% of the students gave the correct answer. However only 57% of the students obtained the correct answer to the second equation. Some of the students (21%) tried to falsely apply the same method from the first equation to the second equation. That is, solving for

$$(3x - 4) = 2 \quad \text{or} \quad (x + 1) = 2. \qquad (13)$$

Their solution was $x = 2$ or $x = 1$. The students failed to comprehend the fundamental reason why $(3x - 4)(x + 1) = 0 \Rightarrow (3x - 4) = 0$ or $(x + 1) = 0$ worked and hence applied it in a wrong context.

3.6.5 Category 5: Judgement in exploring the range of possible solutions

The difficulties experienced in Category 5 occur when students tend to overlook solutions other than the most obvious. Again, a surface learning approach would not contribute to overcoming difficulties in this category. This category does not describe a transition from arithmetic to algebra, a false generalisation or a defect in a control mechanism, but rather a narrowing of vision and a lack of awareness of context. It is this category that comes into effect only after the accumulation of considerable mathematical background knowledge, predominantly based on deep learning of topics. An illustration of this example occurs when solving $\sin x = \frac{1}{2}$. Students may need to consider solutions other than $0 \le x \le \frac{\pi}{2}$. Likewise when students become familiar with complex numbers and functions, they often fail to consider complex solutions as well as real solutions. On the other hand, when graphing a function such as $f(x) = \sqrt{x^2 - 4}$, students can incorrectly use the negative values of $f(x)$. Therefore this category involves more than just looking for all possible solutions, it is about being aware of and investigating all possible relevant solutions. This category therefore involves a defect in higher level judgement skills that include an awareness, investigation and culling of possible solutions.

3.7 Interviews

Analysis of routine algebraic skill tests alone does not give much insight into the cognitive and non-cognitive influences that contribute to these difficulties. Therefore throughout the academic year, as students (approximately N=50) came to see the author, the opportunity was taken to probe deeper into each of the five categories. The following summary comes from comments made by the students:

Category 1:

Ability to apply the order of operations agreement, especially the role of brackets.

Some students

- did not think the brackets were important.
- rote learned the order of operations without meaning but then forgot the order. Their default reaction was to perform the operations from left to right in that order.
- felt they were either not taught to use brackets or taught that brackets could be implied. The students then forgot the implied brackets in the next line of calculation.

Category 2:

Ability to apply the properties of numbers, especially fractions and rational functions.

Some students

- were confused between sum and product in fractions and this extended to rational functions.
- believed they could do fractions when they were taught, but had spent years using their calculators for fractions. They forgot how to add fractions manually.
- did not see the connection between fractions and rational functions. They described a rational function as two separate functions, where one function just happens to be on the denominator.
- felt they never understood fractions anyway.

Category 3

Ability to follow the structure of underlying procedures.

Some students

- concentrated on the details within an equation and did not comprehend the whole equation.
- said that they could not see what to use or do next.
- thought it was acceptable to approximate symbols; for example, shrinking of the divisor line mentioned earlier in the chapter. The students did not realise that they were changing the equation.
- never bothered to check their calculation or did not know how to estimate or judge their calculations.

Category 4

False generalisation.

Some students

- rote-learned a procedure in one context (with little understanding). The procedure was therefore isolated to that context alone.
- saw the similarities and differences between contexts, but did not know how to connect them.
- tried to apply the procedure learned in one context to another when that original procedure was not fully comprehended.

Category 5
Judgement in exploring the range of possible solutions.

Some students

- concentrated on the question and manipulation rather than the solution. The solution itself was not important, only obtaining an answer.
- did not relate the solution to the question. They usually ignored conditions and limitations. Sometimes they did not consider these important, but often they did not know how to interpret the conditions within the question.

Each of the comments mentioned in this section point to an acknowledgement of surface learning as a major reason for why the students displayed the errors. The comments above also portray a multi-dimensional combination of psychological, didactical and epistemological influences (Gagatsis & Christou, 1997). Prerequisite misconceptions and lack of understanding in basic algebra are not likely to be remediated at university where much of the work concentrates on higher level concepts. A major concern is whether these difficulties in algebraic skills that students bring from the schools, remain with students through their university career.

3.8 Summary

This chapter outlines five major categories of algebraic difficulties displayed by first year mathematics students at university. The categories had many features in common with Rotman's ideas in the transition from arithmetic to algebra and appeared to be influenced by a surface learning approach to mathematics. This was reinforced by the comments outlined in Section 3.7. The first category was labelled *ability to apply the order of operations agreement.* This category includes all types of errors that can occur with a particular operation. The difficulty is that the order of operations may not apply in the same way to different algebraic situations. Students who surface learn the order would have difficulty determining when it is appropriate for the rule to be changed. The second category, *ability to apply the properties of numbers, especially fractions and rational numbers,* is similar to Category 1 in that rules in arithmetic may, in certain circumstances, be directly applied to algebra. The problem here is that students often do not recognise the similarity between arithmetic and algebra. It takes more than surface learning a set of basic arithmetic rules to be able to use them appropriately in higher mathematics. Categories 3, 4 and 5 (*ability to follow the structure of underlying procedures, false generalisation, and judgement in exploring the range of possible solutions*) involve higher level skills that emerge from a deep approach to learning. These categories are associated with strategies and planning mechanisms rather than learning procedures.

Each of these categories did not display one type of error but rather a mixture of error types as defined by other researchers. They also involved more than generalised arithmetic skills and often included strategic and judgement skills that are more likely to emerge after students have adopted a deeper approach to learning mathematics.

Chapter 4 The Effect of Higher Learning on Algebraic Skills

The purpose of this research is first to confirm whether the five categories of algebraic difficulties found in Chapter 3 were consistent for the 1997 cohort of first year students. Second, the purpose is to determine whether the same difficulties existed at the end of the first year and in the years prior to, and following, first year university mathematics. This chapter concludes by proposing a possible framework for remediation of these algebraic skills at both the senior secondary and undergraduate levels.

4.1 The tests

In 1997, three sets of algebraic tests were administered to three different levels of students. One (longer) test, administered to first year undergraduate students, was similar to the 1996 tests that determined the categories. This 1997 test was used to confirm the five categories of algebraic difficulties:

1. *Ability to apply the order of operations agreement, especially the role of brackets.*

2. *Ability to apply the properties of numbers, especially fractions and rational functions.*

3. *Ability to follow the structure of the underlying procedures.*

4. *False generalisation.*

5. *Judgement in exploring the range of possible solutions.*

A second set of shorter six-item tests was given to two other cohorts comprising senior secondary mathematics students and second year university mathematics students. The questions used in these short tests are incorporated within this chapter in Section 4.3.1 to Section 4.3.7. Finally, the first year students who sat the longer test at the beginning of the year were given a shorter test near the end of their academic year. These results are discussed in Section 4.3.9.

4.2 Longitudinal confirmation

4.2.1 The test

At the beginning of 1997, first year Bursary mathematics students (N=540) who gained between 50% and 75% in the secondary calculus examination the previous year sat a similar test to the 1996 Group B students. As in 1996, the test was administered early in the academic year (second week of lectures) and was 90 minutes long, but this time there were 28 instead of 36 items. Calculators were not permitted and the students were provided with a table of relevant formulae. Students were encouraged to display all working. A third of these students completed the test within one hour.

4.2.2 The categories confirmed

The analysis of the 1997 algebraic test showed that between 19% and 29% of the students displayed the classic incorrect responses in each of the five categories mentioned in Chapter 3, thereby confirming the five categories found in 1996. Although there were a wide variety of algebraic errors, especially in questions involving integration, differentiation and complex numbers, many of these incorrect responses could be placed into one of the five categories. Only one other possible category was temporarily considered where the difficulty seemed to be limited to non-recognition of notation. Some students (9%) thought that

$$\text{Find } \left|\frac{1}{x+iy}\right|$$

meant finding the absolute value. However, this response was neither widespread nor consistent enough (that is, at least 10% of students) to form a sixth category.

4.3 A comparison between cohorts

4.3.1 The tests

Two senior secondary mathematics classes from a sample of several different schools (N=69) and two second year university mathematics classes (N=142, N=198) sat similar six-item algebraic tests based on the categories determined from the 1996 and 1997 analyses. Calculators were permitted in all these classes and the tests took about 15 minutes to complete. Again, students were encouraged to display their working. Each question in the six-item test was slightly adjusted to the level of the students' mathematical knowledge. For example, a second year university student would be expected to solve $m^2 - 8m + 4 = 0$ as part of their work on second order differential equations, but a senior secondary student would be more familiar with the same question in terms of x rather than m.

A comparison was drawn between the three mathematics levels: senior secondary, first year university and second year university levels. The 1997 first year test scripts were compared with the results from the shorter tests given to senior secondary and second year university students. Data was collected from a random sample of 100 scripts from each of the first and second year classes and all 69 senior secondary scripts. Although the author wished to test each of the five categories with a variety of items in the short tests, the trade-off was for the majority of students to attempt the questions and to limit the tests to 10 - 15 minutes.

4.3.2 Factors affecting comparisons

Several factors could have affected the comparison between academic levels in 1997. At the time, although secondary students were not given any prior warning of the test, they had just completed a month of algebraic revision in formal class time. Year 1 university students had three weeks warning about the test (which was an official part of the course) but any revision was done through independent study. Year 2 students did not have any warning about the test.

4.3.3 Category 1: Ability to apply the order of operations agreement

The following question was chosen as representative of this category and given to senior secondary, Year 1 university and Year 2 university students. Students were asked to simplify the expressions.

Senior secondary	$3x-(\frac{1}{4}y+5z)+1=0$
Year 1 university	$1-(1+\frac{x}{2}-\frac{x^2}{3})=0$
Year 2 university	$6x-2z\frac{\partial z}{\partial x}+3y-(4z-4x\frac{\partial z}{\partial x})=0$

A *correct* answer meant that the students multiplied the (-1) into every item in the brackets correctly and a *no answer* meant that students failed to answer the question without showing any working, and a *distributed property failure* meant that students ignored the brackets by multiplying (-1) with the first item only. *Other, especially rearranging*, meant that students made errors trying to rearrange the variables while keeping the brackets in place.

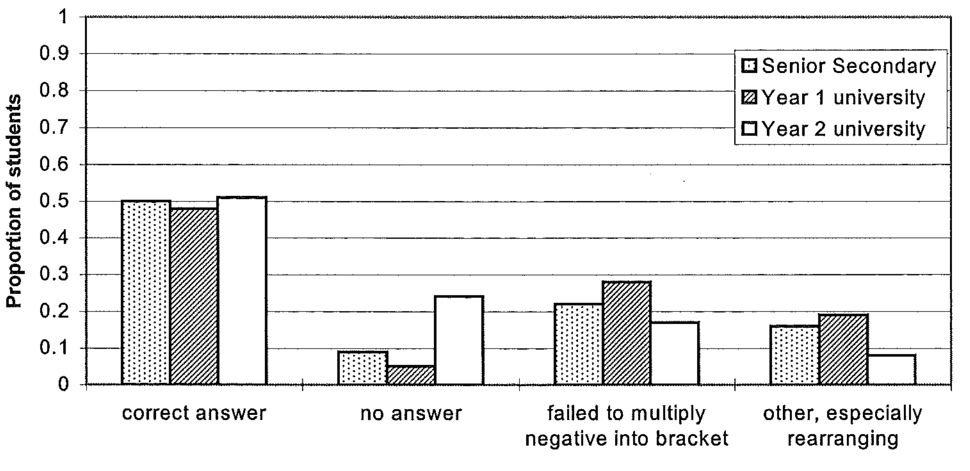


Figure 4.1: Order of operations agreement (brackets)

At the time of the tests, Year 2 university students had just completed a section on partial derivatives, a topic initially introduced the previous year. Figure 4.1 shows that approximately half the students in each of the three cohorts gave the correct answer. On average, 22% of the senior secondary students, 28% of first year students and 17% of second year students ignored the brackets. The slightly lower proportion of Year 2 university students making this same error was expected, as students weak in first year university mathematics were unlikely to pass into second year mathematics. Despite this, almost a quarter of the second year university students did not give an answer or show any working despite having been exposed to partial derivatives at the end of the previous year. Why was there a larger proportion of Year 1 university students than senior secondary students making the same type of error? One possibility is that while the senior secondary students had just completed a month of algebraic revision prior to the test, the university students did not have the same opportunity. The four month break between secondary school and Year 1 university may also have affected the results. What is clear is that the role of brackets used with algebraic elements is a problem before the senior secondary years and that similar errors can persist even with the advancement to higher level mathematics learning.

4.3.4 Category 2: Ability to apply the properties of numbers, especially fractions and rational functions

The following two questions were chosen as being representative of this category. One question required simplification of rational functions and the other involved addition of rational functions.

Senior secondary	Simplify $\dfrac{2x^2-2x}{2x}$, $\quad \dfrac{1}{x-3}+\dfrac{1}{2x}$
Year 1 university	Simplify $\dfrac{ab-a^2b^2}{ab}$, $\quad \dfrac{1}{x+1}+\dfrac{1}{x}$
Year 2 university	Simplify $\dfrac{2n+2n^2}{2n}$, $\quad Y(s)\,(s^2+2s-8)-1=\dfrac{1}{s-3}$ Solve for *Y(s)*.

<u>*Simplification of rational functions (Figure 4.2)*</u>

For the simplification of rational functions, 16% of senior secondary students, 20% of Year 1 university students and 9% of second year university students had difficulty simplifying a given rational function.

As in 1996, *Incorrect A* in Figure 4.2 occurred when students assumed that $2n^2$ was the same as $(2n)^2$. There were slightly more second year university students who gave this type of response where a typical calculation was $\dfrac{2n}{2n}+\dfrac{2n^2}{2n}=1+2n$.

Incorrect B was mentioned earlier, with cancellation of one element on the numerator with a value in the denominator, in such examples as:

$\frac{\cancel{2}n + 2n^2}{\cancel{2n}} = 2n^2$ *or* n. Year 2 university students were slightly better at not displaying this type of error. The addition or subtraction of rational functions are essential elements of some integration techniques in Year 1 university and Laplace Transform in Year 2 university, to name but two applications. The evidence suggests that performance in this category should improve with higher learning.

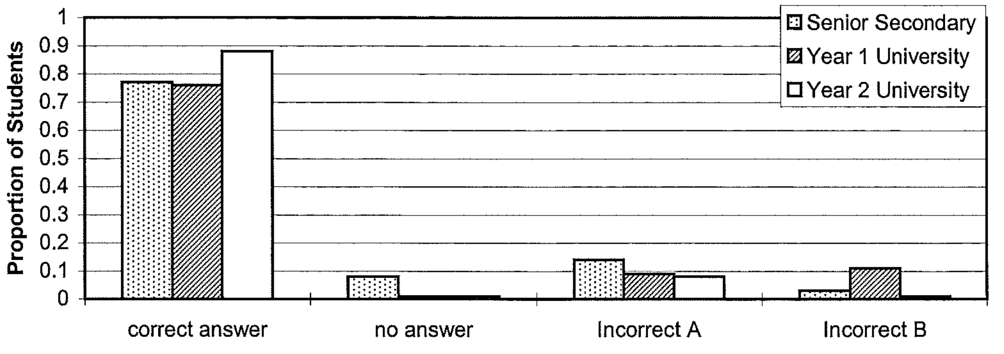


Figure 4.2: Simplifying rational functions

<u>*Addition of rational functions (Figure 4.3)*</u>

The question on addition of rational functions given to Year 2 students seems more difficult, but it was based on a small calculation students encountered just prior to the 1997 cohort test. Lecturers in these classes assumed students could rearrange the equation in terms of Y(s) and add the two rational functions. The results are displayed in Figure 4.3.

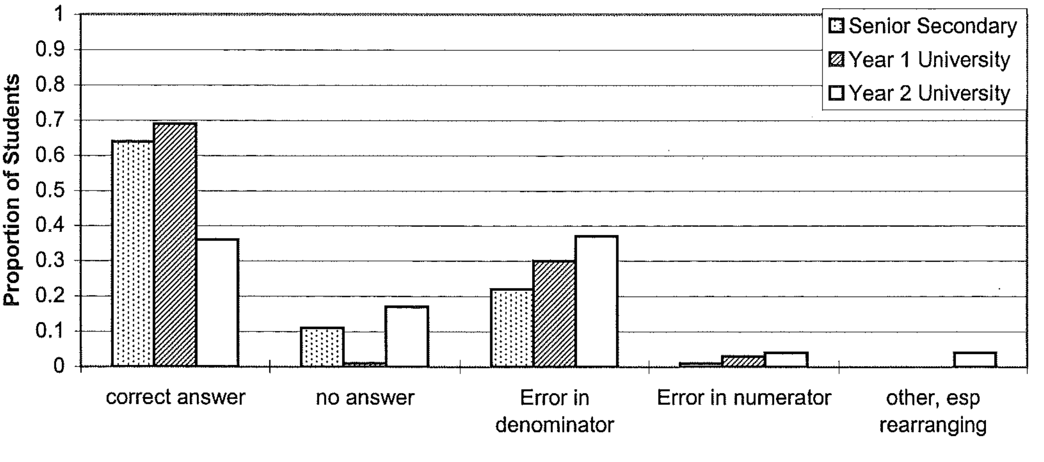


Figure 4.3: Addition of rational functions

All three cohorts displayed difficulty obtaining the correct denominator of the rational function given to them. A quarter of senior secondary, nearly a third of Year 1 university students and just over a third of Year 2 university students could not obtain the correct answer. Even if the denominator was correct, a small proportion of students in each cohort could not then calculate the numerator correctly. In addition, 17% of Year 2 university students and 11% of senior secondary students chose not to attempt the question. Follow-up queries with a sample of these students indicated a choice to skip the question rather than to attempt it.

Overall, simplifying rational functions appears to be handled slightly better by Year 2 university students. Skills involving the addition of rational functions is important for a variety of advanced contexts, such as Laplace Transforms. The results indicate that more students experience difficulty with this type of calculation in the very context in which the skills are needed. It appears that, in general, carrying out operations on rational functions (especially addition) is a skill that does not transfer well to different contexts in more advanced mathematics.

4.3.5 Category 3: Ability to follow the structure of the underlying procedures

In 1996 difficulties in this category were seen in 5% of Group A and 23% of Group B students. Students did not realise that their adjusted equation needed to be equivalent to their original equation. In the 1997 tests, the three cohorts of students could choose between the quadratic formula or completing the perfect square. The questions chosen to represent this category were:

Senior secondary	$x^2 - 8x + 4 = 0$
Year 1 university	$x^2 - 6x - 15 = (x + a)^2 + b$ and $3 - 8x + x^2 = 0$
Year 2 university	$m^2 - 8m + 4 = 0$

The results were similar to the 1996 results. In 1997, 12% of the Year 1 university students who preferred the quadratic formula, but were required to complete the perfect square, exhibited difficulty calculating b for $(x + a)^2 + b = 0$ (Figure 4.4).

Of those students who preferred the quadratic formula, between 14% and 18% of the three cohorts substituted numerical values correctly, but still went astray because of either the gradual shrinking of the divisor line (8% to 12% of students in each cohort) or miscalculation of the value under the square root (5% to 6% of students). In 1996, 13% of Group A and 19% of Group B Year 1 university mathematics students shrank the divisor line in their calculations. This compared with the 14% to 18% in each of the 1997 cohorts. The results appear to be consistent. This type of error displays a lack of mathematical precision with little understanding of equivalence and equality in calculations.

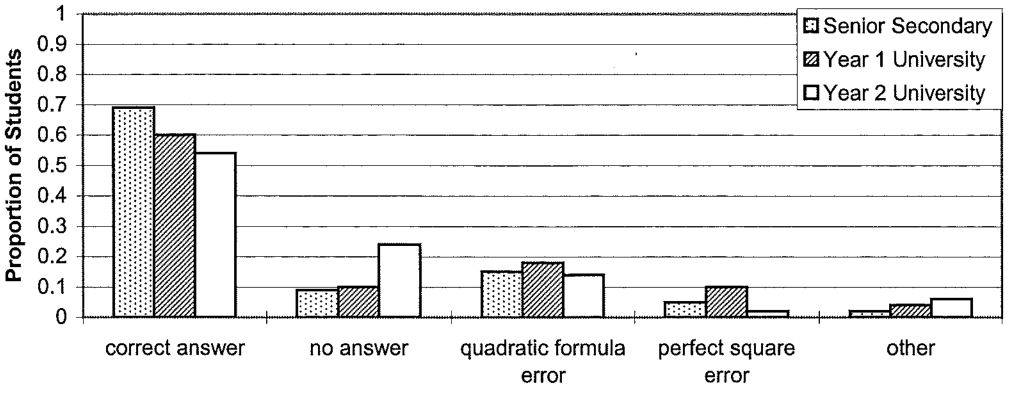


Figure 4.4: The structure of obtaining solutions (solving quadratic equations)

The use of the quadratic formula and completion of the perfect square are only two examples from a huge pool of illustrations for this category. It appears that the percentage of students who have difficulty with the underlying procedures in computation is consistent from year to year and extends through senior secondary, Year 1 university and Year 2 university levels. Therefore the difficulty does not appear to diminish automatically with higher mathematics learning despite the fact that there will have been some attrition and loss of weaker students along the way.

4.3.6 Category 4: False generalisation

In the 1996 analysis of Year 1 students, when the questions were

solve $(3x - 4)(x + 1) = 0$ and *solve* $(3x - 4)(x + 1) = 2$,

93% of Group B students gave the correct answer to the first equation but only 57% obtained the correct answer to the second equation. Some of the students (21%) falsely tried to apply the same method to the second equation as they did to the first equation. That is, solving for $(3x - 4) = 2$ *or* $(x + 1) = 2$. The following questions, similar to those given in the 1996 test were given to the three groups in the 1997 cohort.

Senior secondary	Solve $(x - 2)(x + 3) = -4$
Year 1 university	Solve $(2x - 1)(x - 1) = 5$
Year 2 university	Solve $x(x - 2)(x + 3) = -4x$

Some of the Year 2 university students (29%) cancelled the x's on both sides of the equation and thereby ignored a possible solution of $x = 0$. This type of error is discussed in the final category of incorrect responses.

In Figure 4.5, *assumed the right hand side = 0* means that students ignored the (–4) on the right hand side and calculated the values for x by assuming the equation equalled zero. In 1997, the Year 1 university students exhibited the highest proportion of incorrect responses in this category and most of these (29%) used the same technique for calculating the right hand side of the equation as zero. This compares with the 21% of students giving the same incorrect response in 1996. It should be noted that even though many of the Year 2 university students may have remembered a similar question from their 1996 algebraic test the previous year, 20% of these students did not attempt the question. One possibility is that the extra x variable in the question acted as a distracter or made the question difficult.

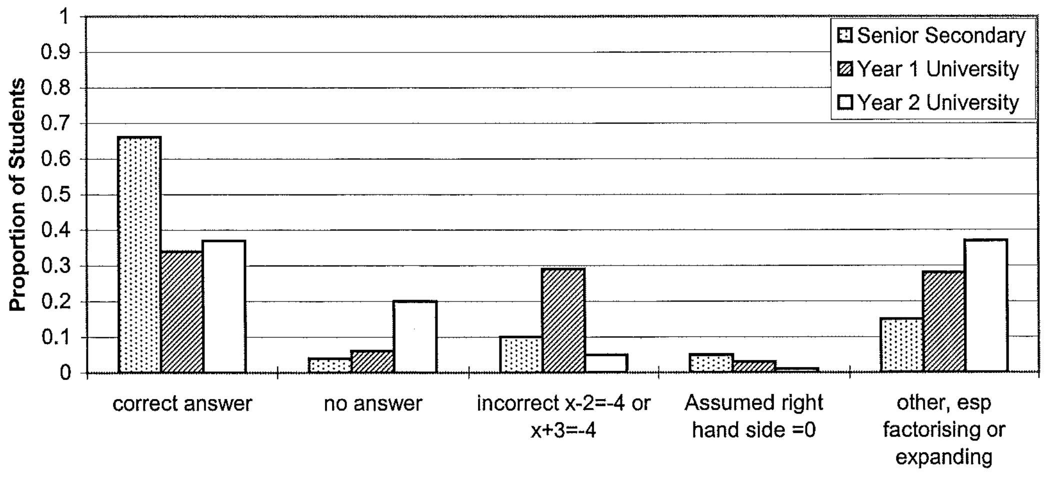


Figure 4.5: False generalisation

At the senior secondary level the teachers commented that they 'drummed' a rule into their students to always leave zero on the right hand side when factorising and then solving an equation. This reinforcement may account for the better performance of senior secondary students on this question. However, those same skills did not carry over to Year 1 for 34% of the university students.

The question designed to illustrate false generalisation in the 1997 tests was more familiar to senior secondary students. The skills to do these particular problems were assumed, but not actively reinforced at university. The evidence points to a decrease in competency, for this category, with higher learning. A side issue that emerged from this category was that factorisation skills appeared to decrease with higher mathematics learning even though factorisation is a skill needed in a variety of contexts in first year university mathematics.

4.3.7 Category 5: Exploring the range of possible solutions

In the 1996 analysis, 28% of Group A Year 1 university students wrote that $x^2 = \frac{1}{3}$ led to $x = \frac{1}{\sqrt{3}}$, omitting a possible negative solution. The percentage of students displaying similar errors may depend on whether the question is framed within a familiar or unfamiliar framework. There were two questions from this category asked of all the 1997 cohort groups. One was a question familiar to all three cohort levels and was asked of all students. The other was a question unfamiliar to all three cohorts that was more complicated and included rational functions and inequalities. In hindsight, this question proved too much of a distracter. The unfamiliar question was asked of all the Year 1 and Year 2 university students and 21 of the more able senior secondary students. For the unfamiliar question the analysis concentrated on whether the students considered both positive and negative values for x and y.

The questions for this category were:

	Familiar	Unfamiliar
Senior secondary Year 1 university Year 2 university	Solve for x if $(x-2)^2 = 9$	If x > y, is it true that $\frac{1}{x} < \frac{1}{y}$?

Most of the students gave the correct answer in the familiar situation, but very few students (between 4% and 6%) considered the possibility of x being positive and y being negative in the unfamiliar situation (Figures 4.6 and 4.7). In Figure 4.7, some students considered negative value answers but did not consider the variables having opposite signs. This group of answers was labelled *considered negative values.*

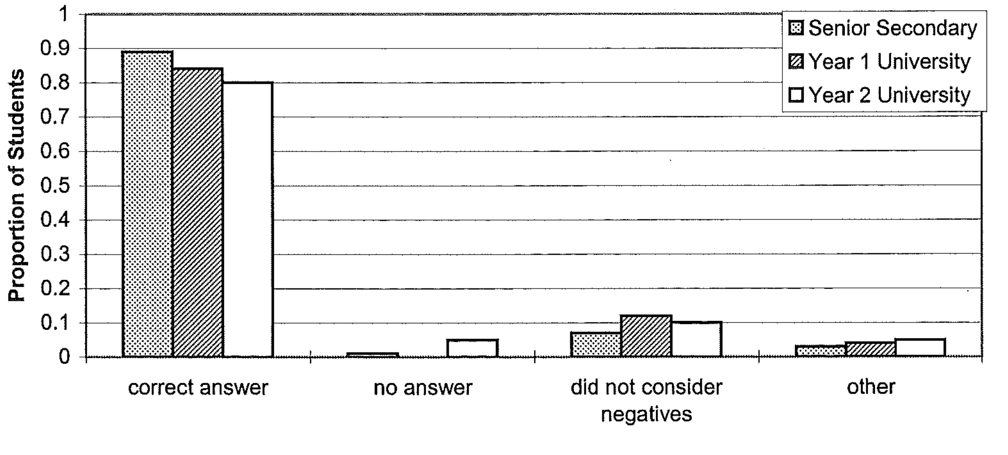


Figure 4.6: Extending the range of possible solutions (familiar situation)

The Year 1 university students were also given a question that involved finding the solutions to a trigonometric equation. The question was similar to that given in the 1996 test and the results were similar in that solutions outside the range $0 \le x \le \frac{\pi}{2}$ were not considered by 20% of the students.

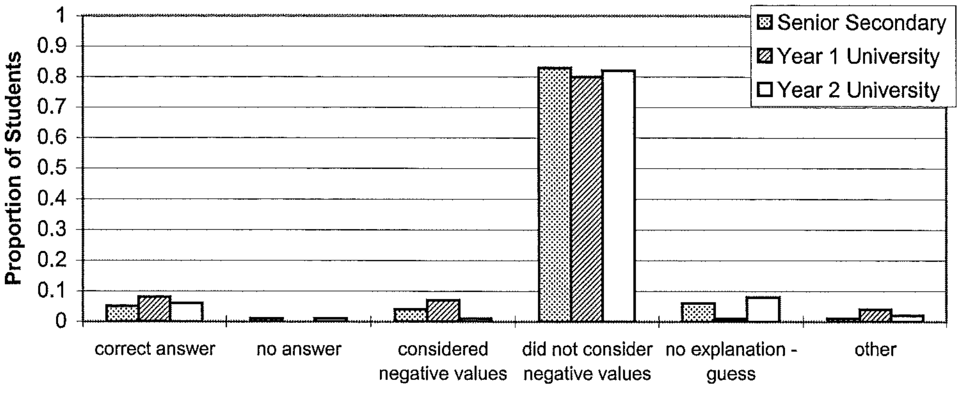


Figure 4.7: Extending the range of possible solutions (unfamiliar situation)

Therefore, although students could be 'trained' to consider other possible solutions within a simple familiar context, this is not necessarily an indicator that students understand or can apply the same ideas to unfamiliar or more complex contexts. The results were similar for all three cohorts, reinforcing the belief that a study of higher level mathematics does not automatically improve student performance of basic algebraic skills in this category.

4.3.8 A Comparison Between the 1997 Cohorts

Some differences and similarities exist between the three cohorts when comparing the proportion of students who obtained correct answers for questions in each of the five categories (Figure 4.8). It should be noted that the question used to identify all *possible solutions in unfamiliar situations* (part of Category 5) could not be adequately compared to the other categories because of the influence of the distracters mentioned in the previous section.

Categories 1, 3, 5 and part of Category 2 (simplifying rational functions) show that a similar proportion of students in each of the three cohorts obtained correct answers. Comparative differences were seen in Category 4 and part of Category 2 (addition of rational functions) where there was some differences between the cohorts. Senior secondary school students performed better than university students in the category entitled *false generalisation* (Category 4). One possible reason was the assurance by secondary teachers that learning to answer this particular type of question was often emphasised in class. The large proportion of Year 2 students who did not obtain the correct answers to part of Category 2 (adding rational functions) was affected by the large number of students who did not attempt to answer the question.

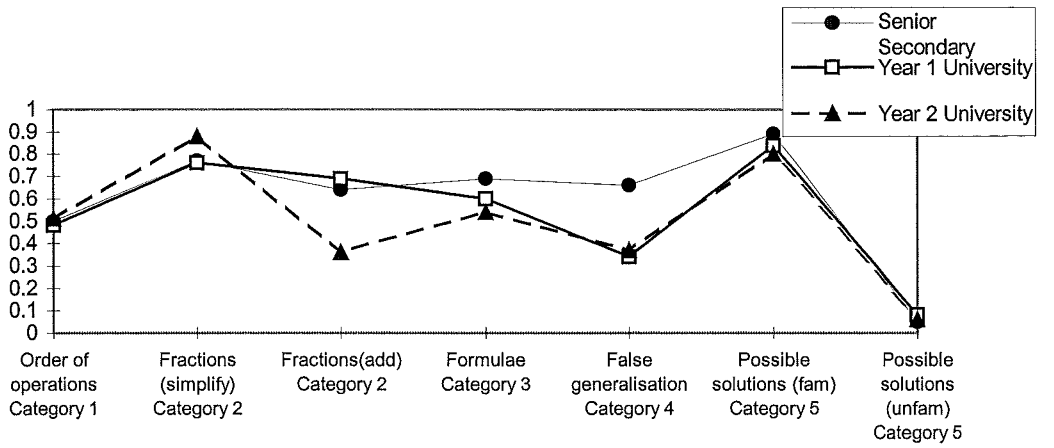


Figure 4.8: A comparison of correct answers for each of the five categories

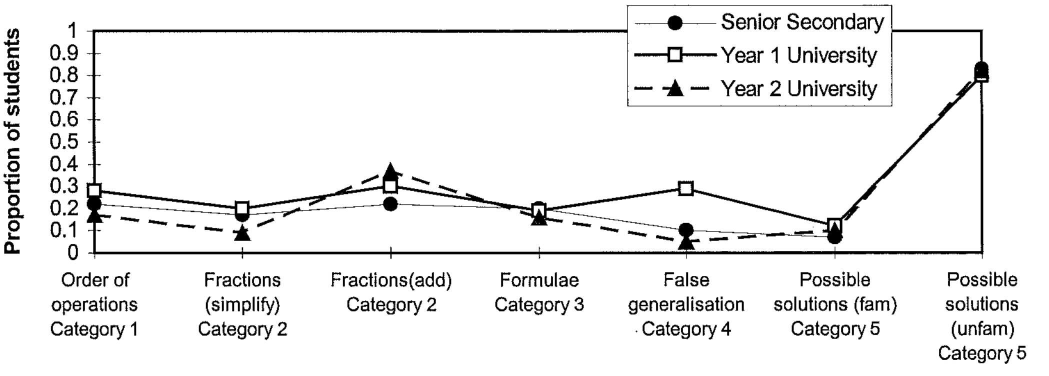


Figure 4.9: A comparison of the same errors for each of the five categories

Figure 4.9 indicates that all three levels, senior secondary and Year 1 and Year 2 university, had between 5% and 38% of students displaying the same algebraic difficulties. In our study, some problems became more apparent by the Year 2 level, such as addition of rational functions (part of Category 2). Other algebraic skills appeared to improve automatically with higher levels of mathematics, such as simplifying rational functions (part of Category 2) and false generalisation (Category 4). In each of the other categories a similar proportion of students displayed consistent algebraic difficulties such as ignoring brackets (Category 1), making errors in formulae use (Category 3) and failing to recognise all possible solutions (part of Category 5).

4.3.9 Improvement during Year 1?

Do algebraic skills improve automatically during the first undergraduate year? Students who sat the 1997 algebraic test early in the academic year had the option of putting their names on their scripts at the end of the academic year. If we refer to the earlier test as Test X and the test at the end of the same year as Test Y, then 185 of the 335 students who sat Test Y chose to write their names on their test script. The results of these 185 students are compared. Their overall group results were not significantly different (95% confidence level) from the remaining 150 anonymous scripts. A difference in test attendance between the original 580 students and the final 335 students could be attributed to course drop-outs or a choice not to attend lectures. The mean scores for these 185 students was not significantly different in the first test (Test X) from the mean Test X scores of the original class of 580 students.

Table 4.1: Mean and standard deviation scores for Year 1 undergraduate mathematics students.

	Test X	Test Y
Number of students	185	185
Mean scores (proportion)	0.629	0.737
Standard deviation	0.156	0.136

There was some variation between Tests X and Y. Test X contained a greater range of algebraic questions and was marked out of 28. The scores were converted to the proportion of marks gained out of 28. Test Y contained seven questions and was marked out of 14. Comparing the two sample means gave a z score of 7.12. Therefore at the 95% level of confidence, the mean score from Test Y was significantly higher than the mean score from Test X. There appeared to be a significant overall improvement in algebraic skills over the first year of university. A comparison was made between categories (Table 4.2, A - G).

Tables 4.2 (A to G) indicate proportion of students for each of the five categories

Table A: (Category 1): Understanding the order of operations agreement

	Test X	Test Y
correct	0.48	0.84
no answer	0.05	0.00
ignored brackets	0.28	0.08
rearranging error	0.19	0.08

Number correct shows significant improvement (t-test score = 10.9)

Table B (Category 2A): Understanding the properties of numbers, especially fractions: Simplification of rational functions

	Test X	Test Y
correct	0.76	0.89
no answer	0.04	0.01
incorrect A	0.09	0.07
incorrect B	0.11	0.03

Number correct shows significant improvement (t-test score = 6.67)

Table C (Category 2B): Understanding the properties of numbers, especially fractions: Addition of rational functions

	Test X	Test Y
correct	0.68	0.93
no answer	0.01	0.01
error in denominator	0.29	0.04
error in numerator	0.02	0.02

Number correct shows significant improvement (t-test score = 16.67)

Table D (Category 3): Understanding the structure of obtaining solutions

	Test X	Test Y
correct	0.60	0.65
no answer	0.09	0.08
manipulation error	0.28	0.26
other (esp fact/exp)	0.03	0.01

Number correct is about the same (not significant) (t-test score = 1.52)

Table E (Category 4): False generalisation

	Test X	Test Y
correct	0.48	0.81
no answer	0.06	0.04
incorrect generalisation	0.29	0.05
assume rhs = 0	0.03	0.05
other (esp fact/exp)	0.28	0.05

Number correct has significant improvement (t-test score = 10.00)

Table F (Category 5A): Understanding and exploring the range of possible solutions: Familiar and easy

	Test X	Test Y
correct	0.84	0.64
no answer	0.00	0.01
did not consider negatives	0.12	0.33
other (rearranging)	0.04	0.02

Number correct shows significant decline (non-improvement)

(t-test score = -6.06)

Table G (Category 5B): Understanding and exploring the range of possible solutions: Unfamiliar and difficult

	Test X	Test Y
correct	0.08	0.03
no answer	0.04	0.04
did not consider negatives	0.80	0.85
other	0.08	0.08

Number correct is about the same (not significant) (t-test score = -1.52)

Over the academic year, algebraic skills for Year 1 university mathematics students improved for Categories 1, 2 and 4, remained the same for Category 3 and the unfamiliar part of Category 5, and declined for the familiar and easier part of Category 5.

Just working through the usual first year mathematics curriculum gives students enough practice for slight improvement in basic algebraic skills such as the use of brackets, dealing with rational functions and having to be selective in choosing the appropriate tools for solving a problem. However, similar proportions of students from the beginning of the year still have difficulty manipulating and using formulas at the end of the year. This is not surprising as much of the first year mathematics assesses understanding of concepts rather than manipulation. The most surprising result was that by the end of Year 1 university, more students had difficulty with looking for all possible solutions, especially in simple algebraic situations. The use of applications limits the option for negative solutions, and this may have influenced the decline in Category 5.

4.4 Remediation - a problem-solving framework?

If major categories of algebraic skills are consistently weak at university the next step is to propose a possible framework for remediation, preferably at the senior secondary and early university levels. By taking a different look at the five categories described here it may be possible to interpret them within a problem-solving framework.

A standardised problem solving strategy has been around for at least 50 years. In the 1940's Polya defined his four basic steps in problem solving as:

- understanding the problem
- planning
- carrying out the calculation
- looking back (Polya, 1945)

Polya's work has stood the test of time and his basic framework can be refined giving his first step of *understanding the problem* as a two-step process.

- defining the problem
- generating solutions (exploring possible alternatives),
- deciding on a course of action (planning),
- implementing the solution (carrying out calculations)
- evaluating the solution (looking back) (Fogler & LeBlanc, 1995)

In 1994, Schoenfeld presented a discussion document advocating the teaching of problem solving in undergraduate mathematics as a way of overcoming difficulties with algebraic skills (Schoenfeld, 1994). He maintained that after 12 years of mathematics, students should have received sufficient content. However what the students displayed was deficiencies in the process of mathematics. He maintained that "when mathematics is meaningful and students are interested in what's going on, the students who need to brush up on their skills seem to do so without much trouble" (p. 60). Schoenfeld advocated that most mathematics can be taught in the style of problem solving and that basic skills could be picked up in meaningful mathematics.

As we have seen in this chapter, this study found that common, widespread basic algebraic difficulties did not, as a rule, improve with higher learning. This also applied to an engineering mathematics class (as part of the Year 2 University group) that used algebra in engineering problem solving situations. The engineering class displayed the same categories of difficulties as the other Year 2 university students. This may disagree with Schoenfeld's assertions that basic skills can be picked up in meaningful mathematics. However, our study tested students at the early stages of engineering mathematics. It is still possible that Schoenfeld's assertions may be correct if problem solving is a major part of mathematics learning over a long period, perhaps from early school years.

Currently, tertiary teaching emphasises higher level concepts and students are unlikely to be explicitly taught basic algebraic skills as part of the mainstream mathematics syllabus. The suggestion here is that students learn problem solving strategies and how to apply them to algebraic situations. This implies an algebraic problem solving approach rather than real-life problem solving.

Suppose we assume that algebraic skills could fit Fogler's five-point problem solving strategy. A link between our categories and Fogler's steps could be:

Table 4.3: Problem solving strategy versus algebraic skills

Defining the problem	What algebraic information is given? What variables need to be found? Are there any patterns and relationships?
Generating solutions	What possible alternative algebraic approaches to the problem are there?
Deciding on a course of action (planning)	What algebraic tools are available? What are the most appropriate ones to use? How should the tools be used? What order?
Implementing the solution	Can the tools be manipulated correctly? Can the solution be simplified correctly?
Evaluating the solution	How reasonable or feasible is the algebraic solution in terms of the problem? Should some of the solutions be discarded? Are there alternative interpretations or answers?

Algebraic skills and problem solving appear to form a symbiotic relationship in mathematics in that algebraic skills become an increasingly integral part of problem solving as mathematical concepts develop. Likewise all steps in algebraic problem solving are dependent upon skills in dealing with algebra.

4.4.1 Remediation at the tertiary and late secondary levels

We could use algebraic skills within meaningful contexts, as Schoenfeld suggested, but emphasise the algebra in problem solving situations. This approach is particularly suited to the tertiary level (or senior secondary level) where students have already acquired 13 years of background knowledge and, hopefully, basic problem solving strategies, from following the objectives of the New Zealand curriculum (Ministry of Education, 1992).

Each of the five categories of common difficulties found in this study is closely linked to different problem solving steps. The studies described in Chapters 3 and 4 support the view that dealing with algebra does not invoke a singular skill or faculty but rather a myriad of predominantly knowledge-based skills (Chaiklin, 1989), especially strategic skills rather than procedural (Table 4.4).

Table 4.4: The relationship between problem solving and the five categories in this study

Problem Solving Steps (Polya, 1945)	Categories from this study	Knowledge Type (Chaiklin, 1989)
understanding the problem	4 and 5	strategic
planning	3 and 4	strategic and procedural
carrying out	1 and 2	procedural
evaluation in terms of context	5	strategic

The knowledge-based skills form links between the five categories in this study and problem solving in terms of understanding a context, choice of appropriate tools, planning and controlling a sequence of calculations, and investigating possible solutions.

There is also a lack of time to teach basic skills at the tertiary level unless students are placed in pre-mainstream remedial or foundation courses. For this proposal to be effective from the algebraic perspective, considerable emphasis would have to be put on the processes and understanding of algebraic skills within a problem solving context. Presenting problems that require both a variety and quantity of basic algebra can do this.

This approach to teaching algebraic skills as a problem solving exercise would require different teaching methods and a different approach to the curriculum. Emphasis would not only be on how to use a process, but also why it is used, how to check and how to interpret all possible solutions. Many of the categories of common difficulties involved strategic knowledge (Table 4.4) and this also needs to be the focus of algebraic skills.

4.5 Summary

Two issues were addressed in this chapter. First, the categories found in Chapter 3 needed to be confirmed or possibly adusted. Repeating a similar test the following year and once more analysing algebraic test results achieved this. For the second consecutive year the same five categories emerged without any further categories or adjustments.

A second aspect of the study was to determine whether these same difficulties were present in senior secondary and later tertiary levels. Using the original five categories a shorter test was devised and given to senior secondary students and Year 2 university mathematics students. The results showed that although there was some variation, any small improvement in the proportion of students making similar errors appeared to be temporary. Overall the same categories of difficulties appeared to already be present in senior secondary students and continued through to Year 2 university. The categories of difficulties therefore did not seem to automatically improve with higher mathematics learning in the long term. There did appear to be some improvement over the first year at university, namely in Categories 1, 2, 4 and 5A as seen in Table 4.2. However, this did not appear to carry to second year at university.

Each of the categories appears to be linked to some step(s) in problem solving. One possible approach to remediating the particular categories found in Chapter 3 is to begin in the schools and teach algebra in a problem solving context. This is an area that requires further investigation at a later date.

Chapter 5 Mathematical Reading Comprehension A review

In secondary school students gain much of their understanding from either teacher/student interactions or using textbooks for routine exercises (as mentioned in Chapter 2). However in university mathematics, the independent reading of expository hard-copy material, whether via textbooks, lecture notes or handouts, currently plays a major role in the self-construction and development of student mathematical understanding. In this way there is a significant jump from secondary to tertiary mathematics.

This chapter gives an outline of the development of reading comprehension. The literature was selected for its appropriateness to this study and specifically focuses on reading models relating to mathematical comprehension for self study purposes. Although the topic of *reading* occupies a large body of research literature, little is found on reading comprehension within the field of mathematics.

5.1 Reading comprehension

Comprehension is defined by Bloom as the ability to grasp the meaning of material. This may include translating words to numbers, interpreting material and predicting. Therefore comprehending goes beyond the simple remembering of material (Bloom, 1968). Reading to learn is the process of comprehending text.

The current research thinking is that learning is more dependent on human cognitive processes such as attention, motivation, metacognition and learning strategies than on the environment. This reinforces a constructivist view that students learn by actively constructing rather than passively receiving knowledge. In accordance with this approach, Wittrock (1990) proposed a model of reading comprehension that combined his work on generative learning with other models of writing (Wittrock, 1974a; Wittrock, 1974b; Wittrock, 1986; Wittrock, 1990). According to Wittrock:

> *...essentially, reading comprehension is the process of actively generating relations among the parts of the text and between the text and one's memories, knowledge and experience.*
> (Wittrock 1990, p. 353)

In this way the learner generates meaning from text. Wittrock had four major components to his model: *generation, motivation, attention* and *memory*. He supported his model with numerous research studies that showed how each of the four components influenced comprehension of text. According to Wittrock, the essential component of comprehension is the process of constructing relations, that is *generation*. The other elements that affect this generation process are *motivation, attention* and *memory*.

5.1.1 Motivation

In order to successfully comprehend text there must be a willingness to invest time and effort in reading. There is evidence that learners who have little motivation in reading or studying a topic experience difficulty comprehending text.

> *Students should become mentally active, generative learners who hold themselves accountable and responsible for constructing verbal and imaginal relations between what they know and what they read* (Wittrock 1990, p.349).

> *The meaning the learners generate about the causes of learning influences their motivation and their willingness to become active in generative learning* (Wittrock 1990, p.350).

Wittrock therefore maintains that the learner must attribute any successful comprehension to his/her own efforts. Motivation as a desire for success is a key element in the approach a student takes to learning (Biggs, 1979).

5.1.2 Attention

Students must be able to attend to the text. When confronted with text the learner needs to be able to filter out irrelevant information, concentrate on the relevant information and to be able to mentally organise and assimilate that relevant information. This may require efficient learning strategies or approaches that become critical when the student has to determine his/her own learning needs (Wilcox, 1996; Young, 1996). For example, work in Sweden and the United Kingdom pointed to different interpretations and approaches to the same text (Saljo 1987; Marton and Saljo, 1976a; Marton and Saljo, 1976b; Biggs, 1979; Ramsden, Beswich et al., 1987; Entwistle and Waterston, 1988). The more accurate comprehension involved deep-level processing that focuses attention on what the author actually means. The implication is that the text writer's exposition or coherence can help or hinder the students' efforts. This is an issue that is supported by McNamara, et al. (1996) and will be addressed later. The less successful learner uses surface-level processing that focuses on the task or text itself (Marton & Saljo, 1976a). Therefore how students attend to a text will influence how or whether they generate the linkages between parts of text.

5.1.3 Memory

The learners' preconceptions (Crawford, Gordon, & Nicholas, 1998), metacognition, everyday experiences and abstract knowledge can influence how they comprehend text. According to Wittrock, *generation* functions by relating content knowledge stored in the memory with the text to be comprehended. Maclellan (1997) defines content knowledge as knowledge about one's physical, social or psychological world. Students without adequate content knowledge could link ideas within a text (that is, what a text *says*) but may be unable to generate the overall meaning by linking the text with memories, knowledge and experience (that is, what a text is *about*). It is this knowledge that helps the reader to decipher what new information is relevant and what is irrelevant. Therefore the general comprehension of text requires not only the linking of ideas within a text but also linking to a learner's experiences, memories and previous knowledge. Influences include the friendliness of the text, and reader resources such as abilities, purposes, perspective, interests and attitudes (Singer & Donlan, 1989).

5.2 Text type and generation of linkages

Text linkages may exist explicitly within the text and/or be constructed by the student. One does not necessarily imply the other in that linkages established by the text author may not be the same as those mentally created by the reader (McNamara et al., 1996).

5.2.1 Narrative versus expository text comprehension

One view is that there are three main types of text. Writer or expressive texts (journals, diaries, personal narratives); reader or persuasive texts (advertising, political documents, editorials, propaganda); subject or expository texts (news articles, reports, summaries, textbooks), (Hoskins, 1986). Narrative text includes the first two categories and the learner is introduced to this type of text when he/she first learns to read. The language and structure of the text is familiar and the concepts often easily relate to the memories, imagination and experiences of the reader (Beck & McKeown, 1989). The end result is a high level of success for narrative reading and consequently a desire by the learner to continue reading.

In comparison, expository text can be problematic and require considerably more effort (Maclellan, 1997). Often the text language is unfamiliar and the ideas alien to the learner's existing knowledge. In expository text the logic is more formal and structured by the author. Consequently, before the learner can focus on the more specialised higher level concepts inherent in the text they first have to learn how to interpret the structure of expository text (Cook & Mayer, 1983). Developing skills to interpret structure is only one of a whole range of skills the learner needs to develop as seen in Figure 5.1 below.

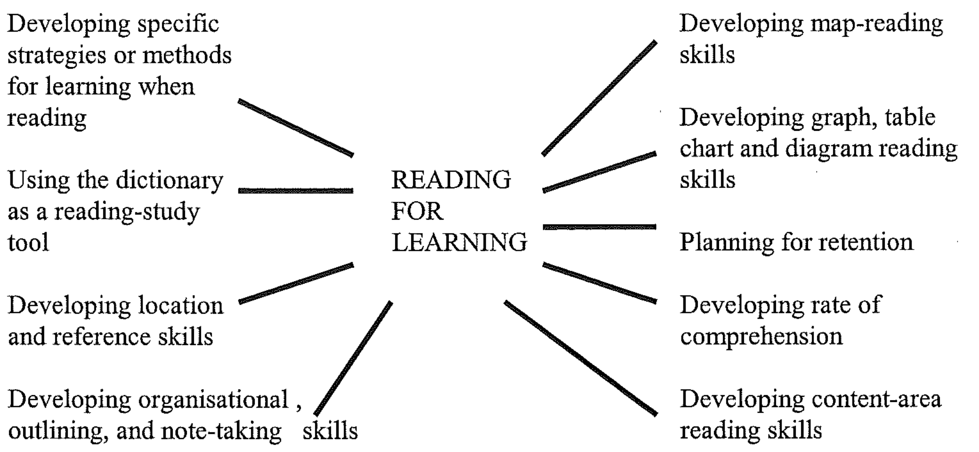


Figure 5.1: Cognitive map: Reading for learning (Dechant, 1991)

In addition to the skills required for narrative text comprehension, reading expository text requires a variety of knowledge structures, such as compare and contrast, problem and solution, question and answer, cause and effect and enumeration (Singer & Donlan, 1989). A range of strategic knowledge and skills is required for reading expository text. Strategic knowledge is about knowing what action to take so that learning can occur (Maclellan, 1997).

Included in the map in Figure 5.1 are skills that use extra resources to supplement reading, selecting, organising and summarising strategies (Mayer, 1996) and the skills to study in specialised content areas. Still other skills needed involve the use of organisational tools such as using pen and paper to make summaries or notes, or perhaps underlining or highlighting sections in the text. Such organisational tools can induce the reader to change their text-processing strategy and to guide the memory-search process (Lorch & Lorch, 1995).

5.2.2 Generation

Unlike narrative reading where the incoming information can be readily linked to existing knowledge, with expository text the learner must mentally develop specific content-area knowledge so that the new information can be linked to that knowledge. The generation of these linkages influences what a learner *selects* from a passage, how they *organise* the information into meaningful units and the degree of *integrating* the new information, that is, making connections between the incoming information and existing knowledge (Mayer, 1996). In this way Mayer supports Wittrock's model of generative comprehension. Mayer also argues that students should be encouraged to learn the appropriate learning strategies for expository text before they master the content.

5.3 Text structure

Kintsch and van Dijk (1978) presented a model for analysing expository text that included arithmetic word problems. They outlined three different levels in the mental representation of text that readers construct:

Surface level:	Processes concerned with parsing text.
Text base level:	Establishment of a coherent representation of the meaning of text (both local and global).
Situational level:	Integration of text content into an individual's knowledge system (that is, linking with prior knowledge).

These levels show a similarity to Mayer's work. Kintsch and van Dijk's levels are not independent of each other and yet they have their own distinct role. The surface level deals with sentences and clauses while the text base level contains the information that is directly expressed and structured in the text by the author. The situational level takes into account that knowledge obtained at a text base level does not necessarily mean that the students have constructed or integrated that knowledge at a deeper level.

The levels can relate to background knowledge and text coherence. For example,

- If there is little textual support, the reader needs strong background knowledge to comprehend at the situational level.
- If the text is too hard, the construction of coherence by the reader may fail.
- If the text is too easy, the gains may be too small as the reader may register satisfactory progress and fail to obtain deeper understanding.

(McNamara, Kintsch, Songer, & Kintsch, 1996)

Although the surface, text base and situational levels give a broad structure of text layers, the researchers acknowledge that expository text comprehension is complicated with disproportionate comprehension at different levels of understanding. In undergraduate mathematics the students' background knowledge is often incoherent and the text is viewed by the student as unfamiliar or 'hard'. Comprehension difficulties arise at all three layers of text outlined by Kintsch and van Dijk (1978).

5.4 Levels of comprehension and understanding

Dechant (1991) identified six hierarchical levels of comprehension and argued that good comprehenders needed to be proficient at the highest possible level. His levels of comprehending were labeled *literal*, *organisational*, *inferential*, *evaluative appreciative* and *integrative.*

In summary form, Dechant's levels were:

Table 5.1: Dechant's six levels of comprehending

Levels of Comprehending	*Processes*
Literal	Recognising and recalling textually explicit, literal or denotative meaning.
Organisational	Recognising the writer's organisation. Converting ideas into a coherent whole. Summarising.
Inferential	Inferring information not specifically stated in text; drawing conclusions.
Evaluative	Making evaluative or critical judgements about the content; inferring cause/effect relationships.
Appreciative	Identifying the mood, tone or imagery, as in poetry, drama, essays.
Integrative	Comprehending for study purposes; to read in the content area.

(Dechant, 1991)

These levels of comprehension imply a hierarchy of achievement from basic recognition of text without meaning to meaning that is both appreciated and integrated with other knowledge. In order to attain each level of comprehension, the previous levels need to be mastered.

5.4.1 Comprehending for study purposes

Dechant specifically includes a level labeled *integrative* comprehending, that is related to study purposes and includes the skills to develop, remember and use or apply concepts. *Integrative* reading has the other higher comprehending levels as prerequisites and as such is considered the highest level of comprehension. This supports the idea that comprehension for study purposes is in a category of its own. For example, Saljo and Marton's (1976a, 1976b) work on deep and surface learning was expanded by Entwistle to include a *strategic* approach. The characteristics of this *strategic* approach were motivational as well as strategic in that they included the intention to obtain the highest possible grades, the organisation of time and distribution of effort to obtain maximum effect, the use of previous exam papers to predict questions and alertness to cues about marking schemes (cited in Entwistle & Waterston, 1988).

As a link to cognitive understanding, Marton et al. (1997) found there were four distinctly different ways of viewing expository learning. These appear to be directly relevant for *reading to learn.*

- learning as committing to memory (words)
- learning as committing to memory (meaning)
- learning as understanding (meaning)
- learning as understanding (phenomenon)

(Marton, Watkins, & Tang, 1997)

Marton and his colleagues were trying to resolve the apparent paradox of Asian study habits that appeared to involve rote learning while resulting in some of the highest comparative scores in international studies. They argued that while a student may commit words to memory without any understanding, the massive quantity of information to remember at university means that the students must grasp some rudimentary meaning of the material in order to select what to commit to memory. Therefore *committing to memory* may or may not involve meaning.

Learning as understanding (*meaning*) implies more permanence. If, having acquired and assimilated that understanding the student is able to do something different with it, then they have achieved *learning as understanding (meaning*). There could be some similarity here to Dechant's inferential and evaluative comprehension levels.

Learning as understanding (phenomena) is appreciating the meaning and being able to relate this meaning with and into other contexts. These ideas are revisited in the next section when we consider mathematical comprehension.

5.5 Mathematical comprehension

There are additional reading difficulties unique to mathematics. The students need to have knowledge of:

- words or groups of words;
- subject or topic knowledge;
- structure of the text;
- how to read (reading strategies).

Mathematical text differs from narrative text in several respects including:

1. Words in mathematics may be used outside mathematics but with a different meaning for example, *integrate*.
2. Mathematical statements generally have a more complicated structure than natural language. Sentences may contain "embedded and subordinate clause structures, complex connectives, and sophisticated word-rearrangement and deletion structures" (Munro 1989, p. 115).
3. Mathematical word statements convey a range of semantic relationships that may be spatial, inclusive, relate to a particular event or change dimension.
4. Mathematical statements are generally context specific.
5. Mathematical statements have a higher density of ideas or concepts and less redundancy.
6. Mathematical statements are more difficult to encode in short-term memory, especially because of word length and complexity.
7. The structure of mathematical text is different from narrative text.

(Munro, 1989)

For many learners in undergraduate mathematics, the jargon and language of symbols is alien to everyday knowledge and experience. In addition, academic learning is focussed on others' views of the real world rather than on the real world itself (Laurillard, 1993). In non-mathematical text, readers can skip words, phrases and even paragraphs and yet still grasp the underlying structure and meaning. In mathematical text, each word, phrase, symbolic expression, numeral, sign, condition, order and position can be critical (MacGregor, 1989). Therefore, the reading behaviour required for expository mathematics text comprehension is non-sequential in that it usually involves skipping backwards and forwards through text, thinking and linking ideas, note-taking and experimenting with calculations. Some of the other difficulties pointed out by MacGregor (1989) include the recognition of mathematical expressions and their equivalent (e.g. $e^{i\theta} \Leftrightarrow \cos\theta + i\sin\theta$), and the tendency in mathematics for a large body of information to be compressed in a short space. An additional difficulty occurs if the meaning does not have an adequate verbal equivalent expression or perhaps even a link to the physical environment.

Most of the literature on reading mathematics has focussed on techniques to help solve word problems or the inadequate structure of textbooks (e.g. Chandler and Brosnan 1994; Chandler 1995; Flanders 1994). Such a focus may be justified as mathematical textbooks in schools are often procedure-orientated with key concepts limited to the introduction. However the contribution of reading to learn in mathematics and its complexity appears to have been underestimated in literature.

Reading mathematics has been interpreted as either a lack of reading skills that are obstacles to learning, or as strategies to teach students to read simple text (Siegel, Borasi et al. 1996; Munro 1989). Other research comments on the nature of mathematical text with arguments for "rich" texts that incorporate the social, cultural and historical dimensions of mathematics (McBride, 1994; Rivers, 1990; Siegel et al., 1996).

Wittrock's generative model of mathematical comprehension and Dechant's *integrative* higher level comprehending for study purposes indicate that reading academic mathematics extends beyond word recognition, word problems and adequacy of textbooks. Mathematical understanding encompasses

> *The comprehension of concepts, the relationships between these concepts and ordinary language or physical concepts. Such comprehension must also include the procedural and process skills which depend upon familiarity with these relationships... Deep mathematical understanding must therefore be primarily relational understanding.*
> (Pirie & Schwarzenberger, 1988, p. 461.)

Table 5.2 contains Dechant's levels of comprehending versus Marton and his colleagues' levels of understanding as mentioned in Section 5.4.1. Marton's ideas are included to illustrate that other approaches, especially more general and cognitive models of learning, reinforce rather than conflict with these comprehension levels.

Table 5.2: Six levels of comprehension versus learning

Levels of Comprehending	*Processes*	(Marton et al., 1997)
Literal	Recognising and recalling textually explicit, literal or denotative meaning	
Organisational	Recognising the writer's organisation. Converting ideas into a coherent whole. Summarising.	Learning as committing words to memory
Inferential	Inferring information not specifically stated in text; drawing conclusions.	Learning as committing meaning to memory
Evaluative	Making evaluative or critical judgements about the content; inferring cause/effect relationships.	Learning as understanding meaning
Appreciative	Identifying the mood, tone or imagery, as in poetry, drama, essays.	Learning as understanding phenomena
Integrative	Comprehending for study purposes; to read in the content area.	

Relevant aspects of this table and a possible link with mathematics reading are discussed later in Chapter 7, Section 7.1.

Arguments for a hierarchy of achievement in mastering mathematical reading have been discussed in literature (for example, Earle, 1976). Earle argued that there were four levels of achievement: perceiving symbols (pronouncing and recognising); attaching literal meaning (clarity of symbol placement); analysing relationships (linking ideas with text and prior knowledge); and solving word problems. The final mastery is dependent on mastering the first three levels. The contention is that study requires a combination of the other levels of achievement in comprehension, perhaps equivalent to Marton et al.'s fourth level of *learning as understanding the phenomena* (Marton et al., 1997). In mathematics, limited success can be achieved by the lowest level of comprehension (*committing words to memory* or literal comprehension), such as rote learning procedures with little understanding of how or why that procedure works (Skemp, 1976). Students may also have limited success even if they understand why a procedure works, but cannot apply the procedure to a different context (*committing meaning to memory*). If a student can use the information to solve problems they have achieved *learning as understanding* the *meaning*.

The hierarchical development of comprehending mathematics from *literal* to *appreciative* or *learning as committing to memory* to *learning as understanding phenomena,* is also reflected in work on the development of comprehending mathematical objects at the University of Warwick in the United Kingdom (Gray and Tall 1994; Tall 1997). Tall and Gray argue that symbols play a dual role: as a process and a result of that process (product). They called this duality a *procept.* These researchers were interested in the way students interpreted symbols in arithmetic, algebra and calculus. They proposed a model explaining how the comprehension of symbols develops, as outlined in Figure 5.2.

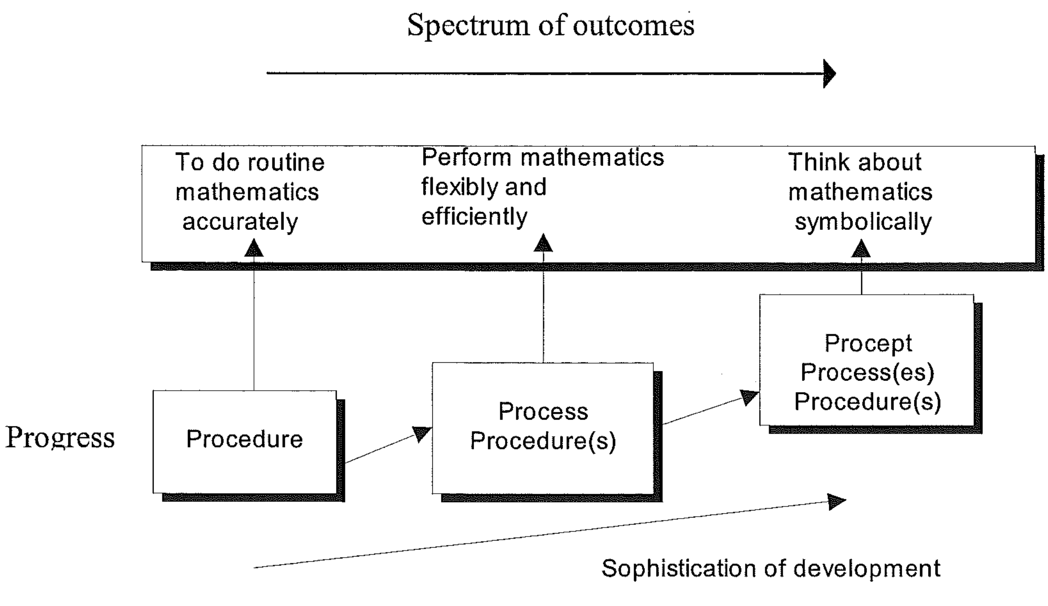


Figure 5.2: Development of the meaning of symbols (Tall, 1997)

Figure 5.2 shows that routine mathematical procedures, such as the addition or subtraction of fractions, can help with recognition of symbols in context. The linking of symbols to the underlying process and to other contexts may help develop efficiency and flexibility in using the symbols. Thinking about mathematics symbolically occurs when the *procepts* are comprehended.

These are symbolic representations that invoke knowledge of both the possible processes involved and the product of those processes. The meaning of the symbol has developed into mental objects where the symbols are seen flexibly as processes to do and concepts to think about. Symbols not only need to be manipulated but also fully understood.

In mathematical text, both the mathematical symbol and phrasing exist. Not only do the symbols need to be comprehended but also the interrelationship between both the symbol and narrative. It is this interrelationship that forms the basis of reading to understand mathematics. At university, students have to develop the skills and strategies to comprehend by themselves content that is unfamiliar to their natural experience and knowledge. They need to generate the interrelationship between symbols and narrative and actively construct the linkages.

5.6 Reading as self-directed learning

Self-directed learning is defined as any process of learning in which the learner functions autonomously, taking responsibility for planning, initiating, and evaluating their own learning efforts (Wilcox, 1996). This has more to do with self management or self direction than learner control (Candy, 1991). Reading to comprehend concepts is only one aspect of self-directed learning that in university mathematics is an essential element for developing content knowledge, critical thinking skills and intellectual autonomy in the field. These critical thinking skills include inference, recognition of assumptions, deduction, interpretation and evaluation, that is, the equivalent of the inferential, evaluative and appreciative levels of comprehending mathematical text.

At the university level textbooks are an important resource for self-study and content information, read with pen and paper in hand (Smith, 1996). Although there has been an initial movement away from textbook reading, the constructivist approach to learning acknowledges the role of text reading to cover content adequately and help students apprehend mathematics structures (Laurillard, 1993).

It is in self-directed learning that Wittrock's model is especially relevant since motivation and attention become key elements. At university we often assume that students have not only developed (or will quickly develop) self-directed learning strategies but that they have the willingness and capacity to comprehend written mathematics. Kreber (1998) found that such an assumption was incorrect. Self-directed learning was welcomed by few learners, in fact only by the more able students who appreciated the choices, flexibility and responsibilities (Kreber, 1998). Kreber's recommendation was that self-directed learning needed to be fostered by teachers in higher education. Recent (1998) on-line internet Mathematics

Association of America articles outline attempts to help university students improve their reading in mathematics. The articles were motivated by a desire for students to read relevant sections of text before attending lectures combined and with a hope that students comprehended text holistically for independent study. Examples of strategies included small groups reading pages of text to each other followed by an analysis of each piece of text in terms of content, form and function; students reading assignments in which students e-mailed answers to comprehension questions prior to each class; or students bringing three in-depth questions from their text to the class for discussion. Current developments in technology may also offer alternative resources for self-study purposes that help overcome motivational problems.

5.7 The role of self-learning technology

New Zealand schools have recently been criticised for not explicitly teaching students to *read to learn* (ERO, 1997). This is especially relevant to undergraduate mathematics where students are expected to master expository as well as narrative descriptive text. The abstract approach of university mathematics texts requires a comprehension level beyond most expository reading found in other fields. The reading behaviour required for expository mathematics text comprehension is also non-sequential in that it usually involves skipping backwards and forwards through text, thinking and linking ideas, note-taking and experimenting with calculations. According to Singer and Donlan (1989) comprehension is also dependent on the friendliness of the text and on reader resources such as prior knowledge, abilities, purposes, perspective, interests and attitudes.

Today, computers are becoming more accessible in the home and school, and this is reflected in research that compares learning with hard-copy typographical text with learning from electronic text, for example, Anderson-Inman, 1995; Anderson-Inman & Horney, 1993; Anderson-Inman & Horney, 1997; Anderson-Inman & Reinking, 1998; Reinking, 1992; Reinking, 1998. Much of the research has outlined the unique features of electronic text including its ability to be modified by the author, to contain multimedia, to have chunks of information linked to other chunks, its ability for information to be easily searched and the advantages for text to be temporarily hidden from view.

The use of computers could also increase the motivation and attention aspects according to Wittrock's model. There is a wide range of literature that portrays the use of computers as a high motivator for students. Students enjoy a course more, they consider computer-based work both relevant and useful, and the students spend more time enjoying problem solving or doing investigative work (Mackie, 1992). Most of the literature also shows that the use of computers in education often results in more than just motivational gains in all curricular areas (Hasselbring, 1986). This is especially so for mathematics, a field considered by many students to be their most difficult topic to comprehend. Computers may make it possible for students to gain insight into mathematical concepts they may not otherwise understand using traditional means (Tall, 1994). Students could retain their knowledge longer (Williams & Zahed, 1996). The use of graphics and animation can contribute to different learning outcomes that make the text more meaningful (Rieber, 1996).

Although many positive learning outcomes through using computers in university have been well documented (Laurillard, 1993), Kulik and Kulik's (1991) meta-analysis of 254 studies found such positive effects in the post-secondary level are small compared to younger groups. These small positive effects were confirmed by Tjaden and Martin (1995) who found that although their 28 computer science students learned the materials faster compared to face-to-face teaching, and the students had more motivation and interest in their tasks, the average student did not significantly benefit, (in terms of grades achieved) from learning via the computer. The groups gaining the most benefit contained either the more able or least able students (academically). Likewise, work by Algama and colleagues (1996) comparing face-to-face teaching, text print and video found that their 72 students who used a video approach scored the lowest on their post-test scores.

Despite the lack of gains in post-test scores, it is not conclusively known whether the current developments in software would help mathematics students improve their comprehension of concepts. In an attempt to support claims for success there has been a move to incorporate, into the programs, many of the positive attributes associated with computer-assisted or computer-based learning. But how successful are they?

The self-study software packages available today may have gains over printed text that include a non-sequential approach of content that allows for more learner choice, the combination of graphics, animation and text, and a user-friendly interface that requires little or no training. A further advantage is the novelty of using a computer package, although there is some evidence to indicate that any such enthusiasm is likely to be a temporary phenomenon in the long term (Lawson, 1995).

5.8 Summary

Comprehension is defined as the ability to grasp the meaning of material. In mathematics this reinforces a constructivist view that the student has to actively comprehend mathematical text.

In this chapter, the focus on mathematical reading comprehension is based on a model developed by Wittrock (1990). As a way of describing how the learner generates meaning from text, Wittrock's model focuses on four components: generation, motivation, attention and memory. The difficulty with mathematics is that it can be considered the most extremely complicated and detailed type of expository text. There appears to be a lack of research exploring mathematical text and comprehension of that text. Some of the literature relevant to this area needs to be extended from research on narrative text. For example, Kintsch and van Dijk outlined three layers of narrative text that they labeled 'surface', 'text based' and 'situational' levels. This points to a possible approach by focusing on layers of text.

Wittrock emphasised 'generation' as the main component in his model with the other three components influencing generative learning. A concentration on generative learning and layers of text means that levels of comprehension in mathematics need be included. The suitable model to assist with levels of comprehension came from a narrative reading model developed by Dechant (1991). Dechant outlined six levels of comprehending that he called 'literal', 'organisational', 'inferential', 'evaluative', 'appreciative' and 'integrative'. These levels of comprehension can also be linked to cognitive models including a recent model by Marton and his colleagues (1997) and to work by Tall (1997). At University, self learning is the dominant strategy for mathematical comprehension and the future may see this self learning including technological developments. Substantial literature relating to mathematical comprehension of text is lacking in these areas.

Chapter 6 Reading to Learn in Undergraduate Mathematics

This chapter describes projects, experiments and interviews that explore the extent to which undergraduate students read mathematics for self-study purposes, and studies the processes the students used. The next chapter (Chapter 7) relates the analysis of this data to a reading comprehension model.

Chapter 6 is divided into three sections. In the first section (Section 6A), we discuss the results of a brief questionnaire which was given to a range of undergraduate mathematics students. The aim was to determine how much time students spent reading mathematics.

In the second section (Section 6B) a reading task was given to a large class of first year students. The aim of this section was to look at the strategies the students used while reading, and the resulting level of comprehension. Both of these sections study the comprehension of a whole topic or part of a topic, generally aiming at a macroscopic view of text comprehension.

The third section (Section 6C) collates a miscellaneous group of small exercises and interviews to give a more detailed look at text layers and a more microscopic view of text comprehension.

Section 6A - The situation

6.1 Do undergraduate students read mathematics?

Considering the emphasis on independent study as a source of developing intellectual autonomy in undergraduate mathematics, it is important to establish whether students are reading mathematics, and if they are, what strategies they use when comprehending mathematical text.

A questionnaire was distributed in lectures to 287 first year mathematics students, 58 second year mathematics students and 69 third year mathematics students. To ensure that the responses were a reflection of a typical academic week, the questionnaire was distributed to the classes when pressure from mathematics tests was at a minimum. The questionnaire asked students for the approximate time they spent (during the previous week) on lecture notes, handouts, reading the textbook to understand concepts, doing practice exercises or answering tutorial and assignment questions. Students were also asked to comment on what they thought of their textbook (or recommended reading). Appendix B contains a copy of the questionnaire.

6.1.1 Demographics

Students involved in this survey had a weekly load of four one hour lectures and a one hour tutorial (15 students per tutorial). Approximately 91%, 80% and 50% were aged 18 to 20 years for Years 1, 2 and 3 respectively. In Year 3, 40% were between 21 and 24 years of age and 12% were mature students over 24 years of age. Mature students comprised only 3% to 4% in Years 1 and 2.

For both first and second year students the ratio of males to females was 3:1. For the third year students the ratio was 4:1. When asked to rate their own ability, 63% first year students rated their ability as average and 23% as above average. In contrast, 55% in the second year and 49% in third year rated their ability as above average, and 22% in second year and 27% in third year rated themselves as average. Females tended to rate their own ability lower than the males in any year as seen in Table 6.1.

Table 6.1: Proportion of males and females for each year with their self-rating

	Self Rating	Below average	Average	Above average or well above average
Year 1	Males	0.09	0.60	0.31
	Females	0.24	0.55	0.21
Year 2	Males	0.00	0.24	0.76
	Females	0.35	0.35	0.30
Year 3	Males	0.12	0.26	0.62
	Females	0.38	0.23	0.38

The return rate on the questionnaires was 91% for Year 1, and 97% for Years 2 and 3. The sample was not random since the students who were surveyed were the ones likely to regularly attend lectures.

6.1.2 Self-study time

Students were asked to comment on the number of hours they spent on mathematics, excluding formal lecture and tutorial hours. Students at all levels were doing full-year courses but while the first year students were surveyed in a 12 point course, the second and third year students were surveyed in a 6 point course. However, most of the second and third year students were also taking 12 points in mathematics so a comparison could be drawn between the different levels. Among the first year students surveyed, one class (N=58) consisted of very able students who had gained high marks in their final secondary school mathematics examination. This class is highlighted with (*) in Table 6.2 as their results were distinct enough to warrant separation.

Table 6.2: Hours spent on study versus year of mathematics study (percentage of students in each year)

%	none	< 2 hours	2 - 4 hours	4 - 6 hours	> 6 hours
First year	6	43	41	5	5
First year*	12	36	50	2	0
Second year	21	31	36	10	2
Third year	39	29	16	10	6

The higher the level of mathematics, the greater the percentage of students who did nothing extra on mathematics self-study outside scheduled lecture and tutorial times. In any year about a third spent up to 2 hours on mathematics self study. Half the very able first year students spent two to four hours on mathematics study as opposed to 16% of third year students. Overall, only a small percentage (10% to 16%) of students spent longer on self-study than they did on formal lectures.

Of the time spent on mathematics outside lectures and tutorials most of it was spent using the textbook for assignments or preparing for tutorials, as seen in Table 6.3.

Table 6.3: Average hours spent on self-study outside formal lectures and tutorials

	lecture notes	textbook for concepts	textbook for exercises	tutorial or assignments	other (help from others)
First year	0.72 hrs	0.23 hrs	0.25 hrs	1.30 hrs	0.00 hrs
First year*	0.45 hrs	0.32 hrs	0.13 hrs	1.22 hrs	0.03 hrs
Second year	0.52 hrs	0.18 hrs	0.10 hrs	1.66 hrs	0.05 hrs
Third year	0.53 hrs	0.47 hrs	0.36 hrs	0.88 hrs	0.05 hrs

For all three years, the average amount of time spent reading to learn mathematics outside set tutorial or assignment tasks was less than one hour per week. Although this was in a normal academic week, time spent on lecture notes and exercises would be expected to increase significantly whenever a test or examination approached. The average first year student appeared to spend comparatively more time re-reading their lecture notes. Second year students used the textbook slightly more for tutorials and assignments and the least for understanding concepts. Third year students were more evenly distributed in terms of time management but spent less time on tutorial/assignments and more time reading their mathematics texts to learn concepts.

Table 6.4: Comments about reading the textbook (percent of students)

	negative	did not use	positive	no comment
First year	37	4	49	10
First year *	29	0	66	5
Second year	41	26	21	12
Third year	14	34	43	9

Reluctance to read set mathematics texts may be seen from the comments students made when asked about their textbooks (and/or recommended reading).

Many students have a negative opinion of mathematical text, including textbooks, recommended reading and lecture notes as illustrated in Table 6.4. First and third year students gave more positive than negative comments, and this coincided with their more frequent use of the textbook for practice exercises. However, the first year positive comments about the textbook as a learning resource were given with reservations:

> *Overall, the resource is quite good, but some of it is too complicated and for* [some] *questions there are not suitable examples for it.*

This indicated a reliance on examples to help with exercises. Other first year students who wanted the answers to everything also implied this reliance:

> *Very annoying how it doesn't have answers for all questions.*

> *Its ok but it would be better with <u>all</u> the answers.*

The most positive comments came from the more able first year students who felt the textbooks were:

> *Good, easy to understand, good examples and exercises.*

However, approximately a third of first and second year students felt that the textbooks recommended for the course were either too difficult to read or did not contribute to their understanding of mathematics. For second year students their comments coincided with their more infrequent use of reading text:

> *Hard to understand - very indepth - more so than needed.*

> *It is not written in easy to understand language.*

> *The wording is quite hard to understand in some places.*

> *The theory is hard to understand to get the main points in a particular topic.*

> *It needs to be more readable. Too many letters etc. I often find myself reading over and over and over the same thing to try and understand it.*

Several comments by third year students included:

> *Hardly ever use it. Mostly for worked examples, although these are frequently too hard, too easy or not similar to what it is I need to understand.*

I would like a lot of worked problems. Best way to learn!

Seemed alright to read earlier in the year, but I haven't used it since.

The majority of third year students did not offer an opinion.

The overall comments indicate that many students did not engage in a great deal of self-study outside set tasks. The various comments indicated a difficulty with being able to read and comprehend mathematics text, and a desire for doing lots of exercises.

These results were confirmed with a large class of first year students in the year following the questionnaire. This time, the students were asked how often they used their textbook to understand concepts as illustrated in Table 6.5.

Table 6.5: Reasons associated with frequency of using textbooks in first year mathematics

	Always	**Often**	**Sometimes**	**Rarely**	**Never**
Percent of all students	**10% (N=38)**	**35% (N=129)**	**30% (N=111)**	**6% (N=22)**	**1% (N=2)**
Lectures hard to understand Text gives better explanation	5% (N=17)	11% (N=40)	9% (N=35)	—	—
Text gives better examples	2% (N=9)	10% (N=38)	—	—	—
Used only for tutorial problems Not used to understand concepts	—	5% (N=16)	5% (N=18)	—	—
Text too difficult to understand	—	—	7% (N=28)	—	—
Felt they did not need to read maths at all	—	—	—	2% (N=7)	1% (N=2)
No explanation given	3% (N=12)	9% (N=35)	8% (N=30)	4% (N=15)	—

Of the 374 students contributing to Table 6.5, just over a third often used their textbook and a third sometimes used their textbook. Within these groups, only 40% of the students who often used their textbook and 35% of the students who sometimes used their textbook did so to understand mathematical concepts. Their motivation for resorting to the textbook appeared to be difficulty understanding the lectures.

Therefore there were consistent difficulties associated with reading text. Many students appeared to open their textbook predominantly as a last option when lectures were too difficult to understand. These students found the textbook easier than lecture notes but additional comments indicated that even they still found reading mathematics difficult.

Section 6B - The process

6.2 How do undergraduate students read mathematics?

First year mathematics students (N=374), were given one of two extracts to read in a 50 minute lecture period. The extracts were taken directly from their textbook and the topics, at the time of the study, had yet to be covered in the course. The purpose of this project was threefold; to determine how students read mathematical text (the process); to determine the areas of ease and difficulty in reading mathematical text; and to ascertain the level of comprehension attained (the product).

6.2.1 *The students*

Approximately half of the 374 were given one extract and the other half were given a second extract. A third of the students were between 19 and 25 years and 60% were either 17 or 18 year of age. The ratio of males to females was 2.5 : 1. Only 15 of the 374 students submitted a blank return, resulting in a 96% return rate. The extracts were given to the students a third of the way into their full year course.

6.2.2 *The extracts*

Relevance and time constraints played a major part in choice of extracts. That is, the extracts needed to contain concepts relevant to the course and needed to be short enough (1 to $1\frac{1}{2}$ pages) for most of the students to comprehend and answer questions in a one hour lecture period. A major reason for choosing the extracts from two different topics, and giving each to half the class, was to check the consistency of the results.

One of the extracts was on *Newton's Method.* This topic was not new to all the students as it can be found in the final year secondary school 'Mathematics with Statistics' syllabus. However, past experience has shown that Newton's Method is not well understood. About half the class were given this extract to read. The second extract chosen was *L'Hopital's Rule.* This topic was expected to be new to the students. The two extracts were distributed alternately so that students sitting adjacent to each other read a different extract.

The extracts were just over a page in length and included an introduction, definition or theorem, worked examples and a mention of situations in which the rule or method was inappropriate (the counter-examples). Appendix C contains copies of the extracts. Attached to each extract is a short questionnaire asking students how they read the extract and which parts they found difficult or easy to comprehend. Also attached to the end of the extract were four open-ended questions relating to content; one was a computational exercise.

Prior knowledge

L'Hopital's rule was new to all but 1% of the students so prior knowledge was related to limits rather than L'Hopital's Rule itself. For Newton's Method, about 43% indicated they had prior knowledge on this topic, usually from school the previous year. This was later verified as some of the responses to the question on Newton's Method included information not explicitly stated in the extract. For example,

> *Not use it when the turning point just touches the curve or two roots are close to each other.*

Observed behaviour during reading

The amount of page turning and observations indicated the extent to which the students immediately went to the questions at the back before reading the extract. Approximately 20% of the students spent most of their time flipping backwards and forwards between the extract and the questions. This meant that the extract was read after attempting the questions. Reading the questions may have helped direct the reading.

6.2.3 The product of reading. How much did the students understand?

Student assessment

How did the students assess their own comprehension of the extract? Of the 374 students, 66% felt they understood their extract, 24% were undecided and 5% indicated that they did not understand their extract as illustrated in Table 6.6.

Table 6.6: Variety of responses to: "I understood the main ideas in the extract"

	Number of students	Percent of students
Strongly agree	41	11
Agree	207	55
Undecided	91	24
Disagree	12	3
Strongly disagree	5	2

The results for Newton's Method and L'Hopital's Rule in Table 6.8 were not significantly different. When students say they comprehend a text or understand reading content, is this a false sense of comprehension? Content questions at the end of the extract were used to assess the degree to which students comprehended the main ideas and details of the extract. Responses were scored according to the criteria set out in Table 6.7.

Table 6.7: Criteria for assessing students' comprehension of topic

Assessment	Criteria	Possible level of Comprehending (from Chapter 5)
0	Either did not attempt or else showed recognition of some elements in the text, but meaning not conveyed.	Literal
1	Summary of main ideas by direct copying from text. Meaning not conveyed.	Organisational
2	Most of the main ideas with meaning conveyed but details absent or sketchy.	Inferential
3	Main ideas comprehended and some details given within context.	Evaluative
4	Main ideas and most or all of the details obviously comprehended. Impression that topic is connected to other knowledge.	Appreciative

The average student showed they comprehended most of the main ideas but not details for L'Hopital's Rule. In contrast, the average student had difficulty comprehending the main ideas in Newton's Method. A more detailed breakdown showing the distribution at the various levels of comprehension is seen in Table 6.8. There was a significant difference between the two extracts (z-score = 4.64).

Table 6.8: Percent of students and level of comprehension based on criteria from Table 6.7

Assessment	L'Hopital's Rule (%)	Newton's Method (%)
0	1 (N=1)	4 (N=7)
1	22 (N=42)	43 (N=80)
2	37 (N=69)	29 (N=54)
3	33 (N=63)	20 (N=37)
4	7 (N=14)	4 (N=7)
*	mean = 2.249; s.d = 0.93; N=189	mean = 1.804; s.d. = 0.93: N=185

Half the students reading the Newton's Method extract displayed little or no understanding of what Newton's Method was about as illustrated in Table 6.8. For example, many missed the idea that Newton's Method involved successive tangent line approximations to a root, found by calculating the intersection of a tangent line and the x-axis. The 50% that did show some meaning in their summary of Newton's Method contrasted with the 75% of students who comprehended the main ideas in L'Hopital's Rule. Because of the apparent advantage that many students were familiar with Newton's Method the result was not expected and there may be a variety of reasons for this.

Compared to L'Hopital's Rule, the way Newton's Method is presented in the extract may have made Newton's method conceptually more difficult. The graph in Newton's Method involves visualisation with movement of successive tangent lines. At first glance the recursive formula is less obvious and the worked examples are numerically more difficult. In addition, 27% of the students reading the Newton's Method extract stated that the iterations in the exercise question took longer than they expected. Therefore they did not have time to complete the last question attached to the extract. A comparison between student self assessment of their own understanding and their actual score resulted in a correlation coefficient of 0.4 for L'Hopital's Rule and 0.26 for Newton's Method. This is a weak positive correlation that is only slightly stronger for L'Hopital's Rule. Therefore while the average student thought they understood the extract fairly well, this did not strongly correlate with their scores. The conceptually more difficult topic gave the weaker correlation.

In general, undergraduate students appear to read mathematical text and think they have understood it. However, the expected level of comprehension differs between the student and lecturer. From the author's perspective the level of comprehension attained by most of the students in the study is considerably lower than desired.

6.2.4 The process of reading. How did the students read the text?

The short questionnaire attached to the extract asked students to indicate what sections of the abstract they read first, what sections they concentrated on, what sections they found difficult or easy and what sections they felt were important for understanding the main ideas. An open-ended question also asked students to explain how they read the extract.

Many of the students mentioned the *speed* of reading as well as what they read and the number of times they re-read sections. Most students (74%), initially skim read their extract once but 53% of these students went back over part or the whole extract more than once. Table 6.9 outlines the reading process and the average level of comprehension score obtained, based on the criteria in Table 6.7. The scores in the table are recorded for the categories of behaviour in which there were more than 15 students (that is, N>4%).

The lowest marks were obtained from students who skim read the extract once and then stopped. This lower level of comprehension could be attributed to narrative skim reading of expository text. Those who then returned to the definitions/theorems gained higher scores that presumably indicated the attainment of higher comprehension. Skim reading an extract and then concentrating on the examples was almost as successful as slowly re-reading each section of the whole extract.

Table 6.9: Percent of students using different reading approaches versus average level of comprehension

Initial reading	Behaviour	Percent of students	Average level of comprehension
Skim [Fast & surface]	once then stopped	20	1.6
	skim again	3	-
	repeat whole more slowly	4	2.1
	return to definitions/theorems	30	2.2
	return to diagram	1	-
	return to examples	12	2.0
Meticulous	re-read each paragraph before continuing to next	13	2.7
Slow and thorough	re-read definition /theorems	5	2.3
	re-read diagrams/steps	1	-
	re-read examples	1	-

Slightly higher comprehension scores were obtained by those students who re-read each section several times and tried thoroughly to comprehend that passage of text before continuing on to the next paragraph. Statistically therefore the meticulous approach was significantly better than any options that involved skim reading. For those who did skim read the extract, repetition of any aspect of the reading was significantly better than reading once. The second largest difference was between those who skim read once and those who skim read then returned to the definitions and/or theorems.

Therefore the two characteristics of relatively more successful expository reading were

- a slow thorough reading of the text
- a concentration on the theorems and definitions.

It was noted that the students who went straight to the content questions and only read parts of the extract scored on average 1.2 out of four marks.

What became clear from the study was that the students did not display any consistent reading pattern for comprehending mathematics. For mathematics reading, the more successful strategies appeared to be deployed by few students.

6.2.5 Exactly what did the students find difficult to comprehend?

The easiest parts for the students, predictably, were the concrete examples. This was confirmed by 89% of students with the L'Hopital's Rule extract and 84% of students with the Newton's Method extract (see Appendix E for more details).

Students struggled with those parts of the extract that contained any abstract elements. For them the hardest section to comprehend was the theorem for L'Hopital's Rule (55% of students) and its explanation (68%), and Newton's formula including how it was developed (68% of students). Some students (39%to 65%) believed that the parts they found most difficult had to be the most important for comprehending the topic and those who obtained the higher scores confirmed this (Appendix E).

6.2.6 How did students comprehend abstract theorems and definitions?

One content related question asked students to describe, in their own words, the formula for either L'Hopital's Rule or Newton's Method. This meant that students needed to find out how the formula fitted into the extract and to read around that formula in order to make sense of it. Basically they needed to understand the entirety of the theorem or definition in the extract. The purpose of this question was to determine how much students comprehended a section of text rather than the whole text. Broad categories of their interpretations are given in Table 6.10

Table 6.10: Students' interpretation of formula (percent of students)

%	L'Hopital's Rule Extract	Newton's Method Extract
Did not answer	16	58*
Incorrect interpretation	11	19
Literal translation from symbol form into word form	53	21
Meaning conveyed	20	2

* Influenced by 27% of the students who did not have time to complete this particular question.

Student interpretation of the formula indicated a range of answers from a non-reply or a literal translation of the symbols into words, to replies that not only conveyed the meaning but embedded the theorem (or definition) into a whole picture. There was a distinct division between responses that conveyed meaning and those that did not convey meaning. Those that conveyed meaning appeared to have begun to grasp the concept and moved beyond mastering processes. However, from the ones that did not convey a reasonably correct meaning, there were both literal translations and incorrect interpretations. For those that did convey meaning, there was a wide range of responses.

Some students showed that they vaguely comprehended the symbolic equation, others gave the formula some meaning in context and still others linked previous knowledge to their interpretation. For both extracts, most of the students gave a literal translation or did not respond to the question as seen in Table 6.10. This appears to reflect a surface approach to their task. The following quotes from the written responses highlight some of the differences in interpretation.

For L'Hopital's Rule, a typical response was:

The limit of the function $f(x)$ over $g(x)$ equals that of limit f dash over g dash.

In contrast, a deeper conceptual response was:

If $\frac{f(x)}{g(x)} = \frac{0}{0}$ and $\frac{f'(x)}{g'(x)}$ is defined on some open interval, and $\frac{f'(x)}{g'(x)}$ has any finite limit or limit that is $+\infty$ or $-\infty$, we can replace $\lim \frac{f(x)}{g(x)}$ with $\lim \frac{f'(x)}{g'(x)}$ using L'Hopital's Rule.

In the first example, there is little indication that the student had any idea of the meaning behind the formula in the theorem. There appeared to be a direct translation from symbol into word form. In the second quote the student showed that s/he understood that one limit could be replaced by another under certain conditions using L'Hopital's Rule.

Similarly, for Newton's Method some students only recognised the elements within the formula:

$f(x)$ is the function
$f'(x)$ is the differentiation function
x_n is the first x

While other students displayed a recognition of the iterative process:

The next value = the previous x minus the y value corresponding to that x value, divided by the y value of the derivative corresponding to that x value.

Despite the recognition of variables and the iterative process, there is little indication that the student knew what the formula represented. In contrast, the following quote displays greater depth of meaning in what the formula was all about.

This formula finds where the tangent line cuts the x-axis. It is the rearrangement of the equation of a line cutting y=0. The equation [itself] is taking an estimated solution for x, dividing the y value ($f(x)$) by $f'(x)$ (its differentiated function) and subtracting this from the original estimated solution (x). When this answer is obtained, you do the same process to it over again until your answer is to the desired decimal place.

Even between these extremes, there appeared to be a range of interpretations with the majority of the interpretations being at the lower to middle end of the comprehension range.

Section 6C - General linkages

6.3 Interviews and exercises

This section discusses a few isolated questions using in-depth interviews with a small sample of first year mathematics students. The purpose of these questions was to reinforce Section 6B and to determine what students link when they first comprehend both familiar and unfamiliar topics. A further four in-depth open-ended interviews on self study habits give a more detailed look at the layers of text that caused difficulty as well as student preferences for symbols, words or graphics.

6.3.1 A familiar topic

Students were given an exercise covering work they had already been exposed to in lectures. Prior to this exercise they had been given the time and opportunity to assimilate at least some of the concepts.

Within the first ten minutes of their tutorial class, 32 first year mathematics students gave written answers to the question outlined in Example 6.1. Before each tutorial students should have attempted several easier questions designated as pre-tutorial work.

Example 6.1

- *Explain what is meant by convergence (as opposed to divergence). Give a full definition and include as many aspects as possible.*
- *What are the following questions asking you to do? How would you approach these questions?*

  Determine whether the following converge:

  (*a*) $\sum_{k=1}^{\infty} k\left(\frac{2}{3}\right)^k$ (*b*) $\sum_{k=2}^{\infty} \frac{1}{k \ln k}$

The topic on sequences and series had been covered during the week prior to the tutorial and the above questions were part of a set tutorial exercise. The students were permitted to use their textbook or lecture notes to answer the questions. Later, but within the same tutorial, students were approached individually to supplement their answers with verbal input.

Results

The students' explanations of *convergence* ranged from a definition taken literally from the textbook to a definition that included divergence as non-convergence and a link to limits. For example,

> *A sequence* $\{a_n\}$ *is said to converge to limit L if given any* $\varepsilon > 0$, *there is a positive integer N such that* $|a_n - L| < \varepsilon$ *for* $n \geq N$. *In this case we write* $\lim\limits_{n \to +\infty} a_n = L$. *A sequence that does not converge to some finite limit is said to diverge.* [Direct transcription from textbook]
>
> *Moving towards a certain fixed point.*
>
> *A sequence is bounded above or below by a finite limit.*
>
> *Diverges: Non-converging; Increases or decreases without bound or oscillates without tending to a limiting value. Converges: Bounded* $\Rightarrow$ *tends towards a limiting value as* $k \to \infty$ *for* $\sum\limits_{k=1}^{\infty} a_k$. *Applies to a sequence and series as a sequence of partial sums.*

This range of answers corresponds to a range of understanding of the word *convergence*. Many of the students could not verbally elaborate further on their answers. Typical answers which showed recognition of some elements without actually conveying meaning were:

> *I don't know why, just a guess.*
>
> *Converges.*
>
> *The ratio test? I think.*
>
> *Don't understand what to do, even what the tests are.*

Some of the partial attempts were often connected to definitions of *convergence* that did not include divergence. Again, this recognised some elements with little meaning. For exercise (a) in Example 6.1 some responses were:

> $\frac{2}{3} + \frac{8}{9} + \frac{24}{27}$ *Need more information.*
>
> $\lim\left(k\left(\frac{2}{3}\right)^k\right)^{\frac{1}{k}} = \lim\left(k^{\frac{1}{k}}\right)\frac{2}{3} = \frac{2}{3}k^{\frac{1}{k}} = ?$, *can't do it - don't understand different tests.*

For exercise (b) responses included:

$$f(x)=\frac{1}{x\ln x} \qquad f'(x)=\frac{-1-\ln x}{(x\ln x)^2}$$

This answer could be the start of checking that $f(x)$ is decreasing by showing that $f'(x)<0$.

Other students, usually the ones who gave more elaborate definitions for the word *convergence,* not only chose a test that would show convergence or divergence, but performed the calculations and final interpretation correctly. These students looked for cues within the series before choosing their tests. In Example 6.1 question (a), many chose the root test because of the power of k. They had linked the position of this variable k to a particular test. Another chose the ratio test for the same exercise because, although they thought they should use the root test, they stated that they were more familiar with the ratio test and decided to try that instead. In comprehending the questions, students tended to look for similar cues that would trigger a particular area of knowledge. Unfortunately some of that prior knowledge was not fully assimilated. For example, a response to Example 6.1 question (b) was:

$$\begin{aligned} \textit{Integral test} &= -(k\ln k)^{-2}k\frac{1}{k}\ln k \\ &= \frac{-\ln k}{(k\ln k)^2}=-\frac{\ln k}{k^2\ln k^2}=-\frac{1}{k^2\ln k} \end{aligned}$$

Although the student recognised the connection between the exercise and the integral test, the student was not successful with performing the integration itself. If anything, the student appears to have made an attempt at differentiation. The different range of answers to the questions reinforced the varying levels of comprehension at the level of the individual word or component.

6.3.2 An unfamiliar topic

The same students (N=32) were also given an exercise on a topic that was unfamiliar to them. They had just completed a section on second order differential equations and had earlier encountered linear independence in a previous section of work on matrices. The question was:

Example 6.2

The Wronskian of two differentiable functions y_1 and y_2 is denoted by $W(y_1, y_2)$ and is defined to be the function

$$W(y_1, y_2) = y_1 y_2' - y_1' y_2 = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

The value of $W(y_1, y_2)$ at a point x is denoted by $W(y_1, y_2)$ (x) or often more simply by $W(x)$. It can be proved that two solutions $y_1 = y_1(x)$ and $y_2 = y_2(x)$ are linearly dependent if and only if $W(x) = 0$ for all x. Equivalently, the functions are linearly independent if and only if $W(x) \neq 0$ for at least one value of x. Use this result to prove that the following solutions of $\frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x) = 0$ are linearly independent.

(a) $y_1 = e^{m_1 x}, y_2 = e^{m_2 x}$ $(m_1 \neq m_2)$

(b) $y_1 = e^{mx}, y_2 = xe^{mx}$

(Anton, 1995)

The question the students were required to answer was:

Do you understand what information is given and what the question is asking you to do. If so, write your interpretation of what it means. If not, what parts are confusing?

Again, this exercise was followed by personal interviews with each of the individual students (N=32). The purpose of the exercise was to determine what linkages they used, both initially and as they progressed through the exercise.

Results

Many of the students read through the entire exercise once. There were three typical responses that, combined with interviews, ended up being similar. These reflected a surface approach to reading for learning. The students were looking for procedures rather than aiming to understand the concepts.

> *I don't know what linearly (in)dependent means but I understand how to do the question. i.e. find any one value for x such that* $e^{m_1 x}\frac{d}{dx}e^{m_1 x} - e^{m_2 x}\frac{d}{dx}e^{m_2 x} \neq 0$.

Although the students stated, when asked later, that they did not understand the meaning of *linearly independent* and *linearly dependent* they knew what procedure to use. They knew to substitute the equations for y_1 and y_2 into the Wronskian formula. When interviewed, the students talked about skim reading the question, linking the equations with y_1 and y_2 in them to the y_1 and y_2 of the formula, and then going back to the middle portion of the narrative to eyeball search for the connection between the word *linear independent* and $\neq 0$. They could not explain why the Wronskian would work. They just pieced a few critical links together.

Other typical responses were:

$$W(y_1, y_2) = W(e^{m_1 x}, e^{m_2 x}) = e^{m_1 x}\frac{d}{dx}e^{m_1 x} - e^{m_2 x}\frac{d}{dx}e^{m_2 x}$$

$$= e^{m_1 x}e^{m_2 x}(m_1 - m_2) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

To be $= 0$, $e^{m_1 x}e^{m_2 x}$ *cannot be zero. Therefore* $m_1 = m_2$ *which it is not.* $\therefore$ *linearly independent.*

And another was:

$$W(y_1, y_2) = W(e^{mx}, xe^{mx}) = e^{mx}mxe^{mx} - me^{mx}xe^{mx} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

$$= e^{2mx}(mx - mx)$$

$$= 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3)$$

[I have inserted the numbers 1, 2 and 3 for later reference.] Like the previous example, these students skim read the information and then immediately substituted the y_1 and y_2 into the formula and began the calculations. The first part of the calculation was typically successful, although several students did not complete line (1). Even more of the students did not correctly use the product rule and their differentiation was incorrect, as in line (2). A typical response in (3) was to leave the answer as zero. The students verbally confirmed that they linked their answer of zero to the $W(x) \neq 0$ from the narrative section, but since this seemed to be a contradiction they just chose to leave it.

A third type of response, although not typical, was to perform the calculations correctly. At first glance it was assumed that these students knew and understood the passage. However, verbal communication showed that the students had done exactly the same as the students in the previous two responses. Their first link, after skim reading the passage, was to link the symbols y_1 and y_2 in the questions with the formula. Then they linked their answer with $W(x) \neq 0$. They stated that they did not understand what the passage was about, they could just do the calculations.

Despite having covered *linear independence* and *determinants* in their topics earlier, none of the students could explain the link between the determinant as zero and linear dependence. Almost all talked about looking for similar symbol cues in the question so that they could substitute it into the formula. Whether the students could do the calculation or not, none of the students obtained a very high comprehension of the passage.

Other less successful individual responses included:

> *No. I cannot understand the wording of the statement/question. It needs to be separated out into steps.*
>
> *No. I find the notation very confusing and I can't follow it. I am not sure of the term linearly independent either. They are also giving a lot of sort of conditions, that I find very hard to follow.*

These students did not find even the simple substitution link between the y values. Students could skim read a passage and successfully perform the calculations but have no idea of the meaning behind the passage. There is a tendency to link symbols to symbols in questions, especially if this means substituting into a formula. Students do not read in depth those parts that they think are too difficult or not relevant.

6.3.3 Case study interviews

At the end of the academic year, after lectures finished but before examinations, four first year students (3 male and 1 female), volunteered to undergo a taped open-ended interview that lasted from 35 minutes to 60 minutes. See Appendix D for a copy of the transcripts. The interviews explored the symbolic, graphic or narrative preferences students had for reading to learn mathematics. Also explored were the different layers of text where the students experienced ease or difficulty in comprehending a topic.

Preferences for self study comprehension

By the time the students enter first year mathematics they are likely to have developed their own preferred way of approaching mathematics. Of the four students interviewed, one preferred a graphic approach, two preferred a symbolic approach and one a narrative/symbolic approach. All four volunteers were slightly older than the average first year student and appeared to be highly motivated. Although this may constitute a bias in sampling, the aim was to explore mathematical reading in depth. Therefore a qualitative rather than a quantitative approach was preferred.

Graphic preference: One student had a definite preference for comprehension through graphics and an aversion to the symbolic and narrative.

> *Student B: Sometimes a good diagram can just crystalise it. You know what I mean? The whole point... and you don't even have to remember because you know what is happening [from the diagram] and you can make it up yourself. And you don't have to memorise it. It is more a concept than rote learning.*
>
> *Student B: Sometimes if the lecturer uses a new symbol I would just write out the words for it. I found that then I would forget what that symbol means real easily and I am only listening while I write it every time the lecturer uses it. Some teachers get so full of the little symbols for everything that you just can't keep up.*

When the student was asked if he tended to mentally create a diagrammatic image of a concept, the reply was:

> *Student B: No. I don't think so. I can never see that there is anything in my mind. It just makes sense to me if I can just jot down a diagram. I can't see it, but if I write it down I can see it. I also read a diagram. It's a concept rather than a picture.*
>
> *Student B: As soon as I open a book, the first thing that hits me is the diagrams. This makes sense to me. There aren't many of these. You can see what it is doing and you can see what it is looking for.*

When the student was asked if he read or concentrated on the narrative sections, the reply was:

> *Student B: No. I just get everything from the diagram. I was disappointed the whole way through the text because there is not enough diagrams.*

Concrete example preference: Two of the students who were interviewed had a definite preference for doing plenty of worked examples and finding patterns within these worked examples. They rarely studied the diagrams and often skipped over the explanations. For them, narrative reading is not as efficient as worked examples. It should be noted that their symbolic preference was for simple symbols in calculations rather than abstract presentations.

> *Student C: I usually look for a worked example...if I can't find an exact or similar example, I see which I can apply to this situation even though it is a different one. And there isn't too much we can't simply plug in.*
>
> *Student A: I have done a limited number of examples [for the topic under discussion], so I haven't got the experience to see it straight away. .. If you do lots of exercises you start seeing it. You don't really get this from just reading.*

Narrative/symbolic preference: One of the four students was willing to read the narrative as well as examples, but still found the symbolic and abstract aspects difficult.

> *Student D: I do try and read to understand it [a topic]. But I still can't get around it. I can sometimes understand what they write about, but the absolute nitty gritty...no.*
>
> *Student D: I concentrate on the examples. I read the rest, especially the definition. Some of it I can understand, but the more general proofs I can't understand. I find I try to read it thoroughly because I want to find out the nuts and bolts of it... it [the textbook] is not user-friendly.*

The main ideas appear to come from narrative reading, but the "nuts and bolts" are not understood in the more abstract proofs and theorems. The student tries to read both but has difficulty with the latter. This same student said he skipped over the diagrams, or just glanced at them. The student did not see diagrams as significantly contributing to the comprehension of a topic.

Different student preferences for study have been recorded in literature. For example, Felder and Silverman (1988), looking at learning and teaching styles in engineering education, found that students' preferences for learning did not necessarily match traditional lecturing and educational. Their categories were sensory versus intuitive, visual versus verbal, inductive versus deductive, active versus reflective, and sequential versus global. The work by Felder and Silverman was reinforced by Solomon in 1992 when he developed an 'Inventory of Learning Styles' at North Carolina State University (Solomon, 1992). Solomon identified four dimensions as: *Processing* (active / reflective), *Perception* (sensing / intuitive), *Input* (visual / verbal) and *Understanding* (sequential / global).

In mathematics, all these dimensions are possible and student preferences could have an impact on the way a student comprehends mathematics. For example, the student above who preferred the visual approach was a reflective rather than an active processor, while the other three 'verbal' students were more active than reflective. For mathematics it would make more sense to further divide the verbal category into narrative and symbolic. Two of the above students preferred symbolic input while one preferred a combination of symbolic and narrative. All four students preferred data and facts to theories and this indicated a perception that was sensory rather than intuitive.

Layers of text

These four interviews indicate that comprehension of mathematical topics may involve several layers of text. The following quotes illustrate four such layers which will be discussed in more detail in the next chapter:

- Understanding of a word, symbol or diagram.

  Student A: I found polar form difficult. I do not know what it is. I had to look again into the textbook and I must admit I can do it... I can put it into polar form but I don't quite understand what the word means.

  Student A: I find it strange. Somehow you take the square root of a negative number, then they develop this theory about complex numbers...

  Student B: A picture helps clear it up for me.

- The link between symbols, diagrams and words.

> *Student A: It was the way you write it down. You know, sigma and the strange little formula that comes afterwards. I did not know what it meant and how to develop the series from there.*

> *Student D: I can't see how the diagram fits in with the symbols.*

- The linking within a discrete section of text.

> *Student D: I can go back and read it, but I can't tell you what the theorem was about. I get quite bogged down. I don't know whether I need to sit down and look at it a lot harder. I don't usually. I just skip over it.*

- The linking of discrete sections to form a whole topic.

> *Student C: In the end, I just look at all the bits and sections and put them together in my mind. It makes sense then.*

These four layers form the basis of the model discussed in Chapter 7.

Influences

Although not specifically sought, a few influences on student preferences for comprehending a topic came from the interviews. These may be hints that outline obstacles in the levels of text discussed in Chapter 7. The influences are divided into three categories.

- What influences the students to read mathematical text?
- What influences the students not to read mathematical text?
- What influences the students to skip over sections of text?

Although there are doubtless other influences, these are just the ones which happened to emerge from the interviews.

What influences the students to read mathematical text?

Students can see the textbook as a last resort:

> *Student C: I learn by getting into problems and doing examples. It forces me to go to look at relevant notes, relevant part of the text.*

Or directed by the syllabus or lecturer:

> *Student D: There are certain things that I work on, and then I think, they must be in this chapter because I have to work on it. But sometimes I can't find it.*

Or the textbook, in particular, is considered a source of 'how to do' recipes.

> *Student A: I found this formula in the book. So I know how to do it now, but again I don't know exactly what is means. I follow the recipe but I don't know what it means.*

What influences the students not to read mathematical text?

One influence is the self acknowledgment of a lack of reading skills for mathematical text:

> *Student C: I am not a good textbook reader.*

Another negative influence is that the textbook is seen as having 'jumps' in explanation that detract from reading.

> *Student D: That is what I try to do when I read through the text, but I usually can't see the path for the trees. I am surprised the text does not explain things more explicitly so that you can understand it.*

> *Student D: I found that the textbook at times can be terrifying. There seems to be big jumps in the working.*

A third influence was that lecturers are seen to give the impression that the work needed to be memorised. The textbook was too much to memorise.

> *Student B: I get the idea we are supposed to memorise everything.*

What influences the students to skip over sections of text?

> *Student C: Because I can do that. And of course if I had gone through in progression and actually done that part it would have given me the next step far more easily. But I just looked and said: 'I can do that', so I went onto the next bit. ... If I had seen the relevance ....*

> *Student A: Usually, if it was like integration I try to make sure I know what I am doing. But this topic is just something we did for a week. I am OK with just knowing the recipe and not understanding what I am doing. Because we just touched on it.*

Therefore if the student thinks s/he is familiar with the text, that part can be skipped over in reading. This reinforced work on text coherence by McNamara and co-workers (1996). Also, if the student feels it is not an essential (examinable) part, judged by the time spent in lectures on it, the student concentrates on what s/he thinks are the bare essentials, namely learning a formula or recipe. Total comprehension of the topic is inhibited in both cases.

6.4 Summary

A questionnaire survey found that students generally do not use their textbook for trying to understand mathematical concepts. The textbook is predominantly used for set tutorial questions and assignments. Few of the students spent more than one hour per week studying mathematics outside formal lecture times. In addition, many students had a negative opinion of mathematical text in general and this included lecture notes and recommended reading in addition to textbooks. These results were confirmed by student comments.

A second part of this chapter then investigated how students read mathematical text, that is, the process students used. A large group of first year mathematics students were given one of two reading extracts that had a short questionnaire and comprehension test attached to it. It was found that most students skimmed the text as if it was narrative text. The few students who concentrated on the theorems and definitions using a slow thorough reading strategy achieved the high marks in the test questions. These students displayed a deeper comprehension of text. However, the majority of students scored low on the test questions and their description of how they read their extract indicated a surface approach to reading.

These conclusions were reinforced with 32 short interviews during tutorials and in-depth interviews with four first year students. It was found that students had difficulty with the symbolic aspects of reading. One student preferred to study the diagrams as an aid to understanding symbols while most students gave up trying to understand symbols. This often led to theorems being skipped over entirely. Comments from students pointed to a preference for doing examples as a way of understanding a mathematical topic. Again, this emphasised a preference for a surface approach to text reading and a consequence lack of concept recognition.

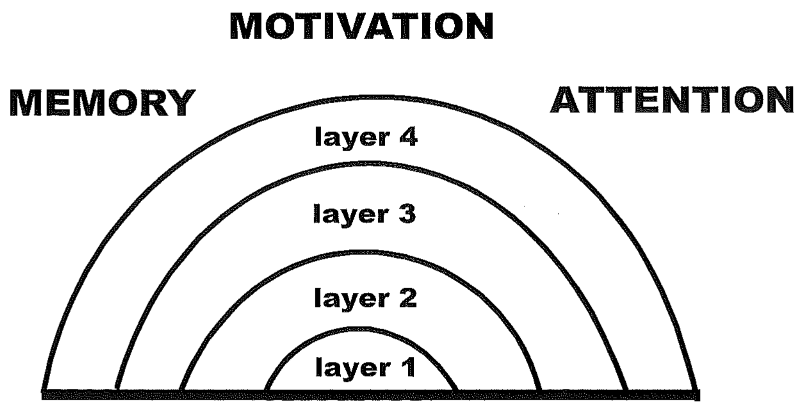
Chapter 7 Linkages in Mathematical Comprehension

This chapter outlines a possible model of mathematical comprehension based on linkages between components within text. It draws on established reading and comprehension models by Wittrock and Dechant and concurs with Munro's (1989) assertion that contemporary reading models can be used to analyse mathematics reading. The model in this chapter emerged during the analysis of the various projects outlined in the previous chapter. The emphasis is on the levels of text and where we need to draw our students' attention for reading undergraduate mathematics.

7.1 Four layers of text and six levels of comprehension

As pointed out in Chapters 5 and 6, mathematical text differs from narrative text, not only for its expository nature but also for its distinctive language (jargon), symbolic components, diagrammatic representations and abstractness. If any two students can comprehend the same narrative text differently (Marton & Saljo, 1976a; Marton & Saljo, 1976b) then comprehending mathematical text must be even more complex since there are more basic components that lead to exponentially more possible distinct linkages between components.

Suppose a section of text describes a main concept. Although we can talk about Dechant's six levels of comprehension of that text, we are often referring to the comprehension of the whole text. Based on the projects outlined in the previous chapter it is proposed here that this comprehension is only the surface of mathematical text comprehension, that is, the outer layer of text. Significant contributions to overall mathematical comprehension come from the linkages at sub-layers of text.



Each layer contains 6 levels of comprehension

Layer 1: 3 basic components
Layer 2: Simple linkages between components
Layer 3: Linkages between sections
Layer 4: Overall text comprehension

Figure 7.1 A Model of Mathematical Comprehension

Figure 7.1 outlines a possible model illustrating four distinct layers of text. At the basic ("microscopic") layer 1, there are three distinct components: *words, symbols* and *graphics*, and each of these components can be understood at any one of Dechant's six levels of comprehension.

Simple linkages between any of these three basic components can also be described in terms of the six levels of comprehension and it is these simple linkages that comprise layer 2. For a more macroscopic view, a section of text (such as a theorem, definition or worked example) can be understood at any comprehension level, while consisting of a network of simple component linkages, themselves comprehended at different levels. Therefore layer 3 represents the small paragraph, section or portion of text. Finally, the macro linkages between the various sections such as definitions, theorems and worked examples are the fourth layer (or outer layer) of text. This layer 4 encompasses each of the sub-layers 1, 2 and 3. Comprehension of layer 4 is therefore the overall comprehension of a topic or concept.

Layers 1 and 2 differ fundamentally from layers 3 and 4 in simplicity versus complexity. Layers 1 and 2 are direct comprehension of individual components or simple linkages between components. The complexity in layers 3 and 4 results from the network of components and linkages arising from layers 1 and 2. The additional existence of dominant or critical linkages could contribute significantly to overall comprehension in layers 2, 3 or 4. The existence of critical or dominant linkages may explain how someone could have a good understanding at layer 4 without necessarily having full understanding at the lower layers. These critical linkages may need to be comprehended at a high level for overall adequate comprehension.

Although this study concentrates on the linkages within text, memory, attention and motivation as described by Wittrock (1990) can influence each layer. These influences are in turn connected to other factors such as concrete experience, cultural and social environments. Cognitive, pedagogical, affective and metacognitive influences have been well documented and are not the focus of this present study.

7.2 The six levels of mathematical comprehension

Before considering the text layers in a little more detail, it may be prudent to revisit the levels of comprehension as a guide to the quality within each layer. In Table 7.1, I have adapted Dechant's levels to mathematical text. These levels of comprehension were originally mentioned in Chapter 5, Section 5.4, Table 5.1.

Dechant's six hierarchical levels equate to six levels of comprehending in mathematics. In this way it may be possible to measure the quality of linkages made by a student.

The adjustment to mathematical text is in both the definition of comprehension and description of levels. We define *comprehension* as reading to make sense of text for study purposes. In this way we assume student output, both verbal and written is a window into student comprehension.

Table 7.1 : Dechant's six levels of comprehension adapted to mathematical text.

Level	**Category** (Dechant, 1991)	**Description** (adapted for mathematical text)
1	Literal	Recognising words, symbols in explicit or literal form. (No meaning involved.)
2	Organisational	Recognising the main procedure. For example, substituting values into a formula or following a sequence of steps or instructions. (No meaning involved.)
3	Inferential	Main ideas have meaning. Comprehending some obvious details and conditions within context. Apprehending the main process (Meaning within context.)
4	Evaluative	Recognising and comprehending out of context; Simple linkage with existing or other knowledge: Different ways of seeing the same thing; (Meaning outside context)
5	Appreciative	Apprehending as a whole phenomenon (entity) in social and cultural context and within the world of mathematics. (Complex linking with wide range of schema)
6*	Integrative	Being able to use judgement skills; Choice; Own preferences. Problem solving as in choosing the most appropriate path/method.

Level 6* was labelled by Dechant (1991) as "Integrative" and defined as comprehending for study purposes. Ideally, to be fully "Integrative", the learner must not only appreciate the entirety of the concepts as a phenomenon, but also have a personal preference. However 'for study purposes' is an ambiguous phrase in our context. The following outlines the possibilities in mathematics that may occur when a learner is confronted by a problem solving situation that involves the selection and use of a formula. A student may study using any of the levels outlined below.

<u>*Literal*</u>: The learner may guess the formula using a trial and error approach. Often the formula is incomplete or incorrect. If correct it was likely to be rote learned without meaning. Their preference for the formula is based on guesswork.

<u>*Organisational*</u>: The learner may take an 'educated guess' based on some familiar or seemingly familiar cues. The learner can choose from a variety of formulas all learned without meaning. Preference is therefore based on familiar cues.

Inferential: The learner may have some rudimentary understanding of the problem, can select an appropriate formula and use it effectively with meaning. Preference is based on familiar knowledge.

Evaluative: The learner can prioritise possible formulas relevant to the problem. This understanding allows the learner to know which formulas are appropriate and which are not. They can also use the formulas effectively. Preference is based on a wide range of possible knowledge and a linkage to that knowledge.

Appreciative: The learner can deduce a range of possible alternatives based on how they see the problem in the context of the whole range of their knowledge. In this way, they see why some formulas are appropriate and why some are not. They can see other ways of solving the problem without using the traditional formulas. Preference is based on choice of options.

Integrative: The learner not only sees the problem and solutions as a phenomenon, they also have personal preferences as to their choice. They develop value judgements and may favour a geometric or algebraic approach. Preferences are based on a combination of knowledge and wisdom.

7.2.1 A link to procepts, reification or encapsulation of an object

Dechant's integrative comprehension is possible if the student has already progressed through literal, organisational, inferential, evaluative and appreciative comprehension. In many ways this development in comprehension can be related to Sfard's reification of a concept into mental objects, Tall and Gray's procepts as symbolic encapsulation of objects, Rybach's encapsulation theory and Dubinsky's reflective abstraction approach (Dubinsky, 1991; Gray & Tall, 1994; Rybash, Hoyer, & Roodin, 1986; Sfard & Linchevski, 1994; Tall, 1997). Each of these researchers talks about a mathematical concept (such as a function, derivative or fraction) being developed to a whole entity in its own right. Not only are the processes comprehended within and outside a context, but they are also seen as an integrated whole.

Although the researchers did not apply their ideas to sections of text, there is a structural similarity with the hierarchical progression outlined in this chapter. Similarly, in a problem solving approach Davis (1984) used the term *integrated sequence* to describe a series of steps or procedures that are eventually seen as a whole. Therefore integrative comprehension refers to the ultimate comprehension when the component, section, or topic of text becomes an encapsulated object or is seen as a phenomenon. The contention is that

> *...as mathematical maturity develops, so does the number of available mathematical objects increase.*
> (Tall, Thomas, Davis, Gray, & Simpson, 1998)

7.3 The four layers of text and quality of comprehension

Mathematical comprehension is more than just the reading of symbols. The narrative and diagrammatic elements combine with the symbolic elements to significantly contribute to overall mathematical comprehension in higher level mathematics. It is this combined comprehension that contributes to mathematical autonomy at the undergraduate level. The six levels of comprehension mentioned in the previous section relate to the *quality* of comprehension of:

- Basic components in a text.
- The linkages between those components.
- The network of linkages and components.

7.3.1 Layer 1 - Basic components

There are three basic components in mathematical text: *words, symbols* and *graphics*. A *word* is defined here as an element accepted in both narrative and mathematical language, such as 'relationship', 'prove', 'simplify', 'exponential'. A *symbol,* generally defined as "a mark or character taken as the conventional sign of some object, idea, function or process" (Fowler & Fowler, 1995), in this context refers to mathematical notation alien to the narrative language. Such examples are $\sum$, Δ , or ∞ . A *graphic* is a pictorial representation, either static or animated.

For undergraduate students, the comprehension of words, symbols and graphics encountered originally in primary and secondary schools is expected to be at a higher level of comprehension than for any new words, symbols and graphic components encountered in undergraduate mathematics. It needs to be remembered that the comprehension of components may also depend on a variety of factors such as context, content, background knowledge, current topic of study, retention, influence of lectures, comprehension of textbook and motivation, to name but a few.

To illustrate the distinctive range of comprehension for one component let us return to Chapter 6, Example 6.1. This is a question that was given to first year students in tutorials. This time, we look at the same example in terms of levels of comprehension.

<u>Literal</u>: (N=4)

> *A sequence $\{a_n\}$ is said to converge to limit L if given any $\varepsilon > 0$, there is a positive integer N such that $|a_n - L| < \varepsilon$ for $n \geq N$. In this case we write $\lim_{n\to+\infty} a_n = L$. A sequence that does not converge to some finite limit is said to diverge.*

The word 'convergence' is recognised, but the definition given by the students is repeated word for word from the textbook. There is no indication that any meaning is conveyed. By taking the literal expression given in the textbook, the definition was limited to sequences without mention of partial sums as a sequence. This lack of understanding was reinforced when none of the four students attempted the exercises. Therefore assigning this answer the lowest level of comprehension fits in with the values usually assigned to such students.

<u>*Organisational:*</u> (N=4)

> *A sequence is bounded above or below by a finite limit.*

The focus was on sequences without any indication of how the idea of series fitted in. This definition was centred only on the word convergence in a limited sense and was not a full explanation. The fact that four students from different tutorial groups gave an almost identical phrase indicates the repetition of a phrase used elsewhere that may have had some meaning in a specific context in lectures but its meaning is not obvious in this statement. Although three of the four students attempted the exercise their attempts appeared to be random and unsuccessful. For example, for $\sum_{k=1}^{\infty} k\left(\frac{2}{3}\right)^k$ a solution was:

> *Set* $f(x) = x\left(\frac{2}{3}\right)^x$ *and log both sides.*

<u>*Inferential*</u> (N=7)

> *The terms of a sequence/series gets closer and closer to a finite value.*

The students included series as well as sequences and displayed some meaning by using the phrase "closer and closer to a finite value". They concentrated on convergence but have not considered divergence as non-convergence. For the exercises one student was partially successful and the remaining six students were unsuccessful.

<u>*Evaluative*</u>: (N=14)

> *converge = the limit at infinity tends (or gets closer and closer) to a finite value.*
> *diverge = if the limit at infinity does not converge to some finite number.*

The students gave the impression they understand the idea of convergence as a limit approaching infinity and have considered the role of divergence. Of the 14 students, two completed the exercises successfully and the rest were partially successful.

<u>*Appreciative*</u> (N=2)

> *Converges means that the limit of a sequence, or sequence of partial sums of a series approach a finite value. Diverges means that the sequence of partial sums go to* $+\infty$, $-\infty$ *or are oscillating.*

The students comprehend the definition in more detail. For sequence/series it is the idea that the sequence of partial sums approaches a finite value. These students not only show an understanding of convergence and how divergence is non-convergence, but they also recognise the series as a sequence of partial sums. Both students successfully completed the exercises.

<u>*Integrative*</u> (N=1)

> *Diverges: Non-converging; Increases or decreases without bound or oscillates without tending to a limiting value. Converges: Bounded* $\Rightarrow$ *tends towards a limiting value as* $k \to \infty$ *for* $\sum_{k=1}^{\infty} a_k$. *Applies to a sequence and series as a sequence of partial sums.*

In this example, the student displays additional understanding by linking the limit to infinity with the symbolic form for a series. The student knows to apply the definition to sequences and series as a sequence of partial sums. That the above response should be placed in this category was reinforced by the student's correct solutions to the exercises.

These different interpretations of the same word, *convergence*, reinforce the assertion that a basic component can be interpreted at any level in a hierarchical range of comprehension levels. Some validation of these comprehension levels was the increasing level of competence with calculations. Further work is required to find definite correlations between levels of competence and levels of comprehension.

As seen in Figure 5.2, Tall and Gray refer to a three-step hierarchy of procedures, processes and procepts before students think about mathematics symbolically. The approach taken in this study does not contradict Tall's model but rather says that, based on observations and analyses, the formation of procepts requires a high level of comprehension. For any component, there also appears to be more than three steps.

7.3.2 Layer 2 - Simple linkages between components

In mathematical comprehension words, symbols and graphics rarely stand alone. A typical mathematical text has a complicated network of simple linkages between each component. In this study, the term "linkage" refers to the association between components that makes sense mathematically. In this study, the linkages between the three components and the quality of those linkages is of prime importance.

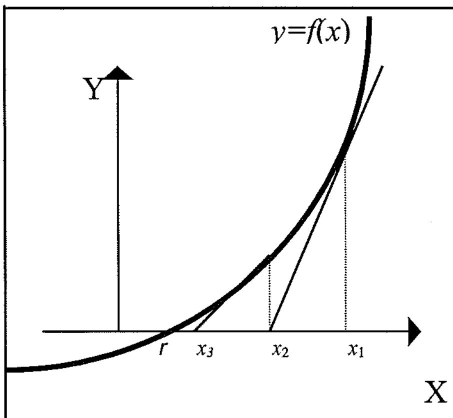
One component can link to any other component within text. Ideally this should only occur if the link makes sense, but surface learning can lead to ill-conceived linkages that do not make sense mathematically or narratively. We saw this illustrated with algebraic cancellation of rational functions in Chapter 2 and from the extract data in Chapter 6 (& Appendix E.)

To simplify these connections, six possible types of linkages can be identified as:

word ↔ word (WW)	word ↔ graphic (WG)
symbol ↔ symbol (SS)	symbol ↔ graphic (SG)
word ↔ symbol (WS)	graphic ↔ graphic (GG)

The following extract is taken from a typical first year textbook (Anton, 1995) and describes the basic ideas of Newton's Method. (Example 7.1 is selected from Newton's Rule abstract in Appendix C.)

Example 7.1: Newton's Method for finding roots of an equation



Suppose that $x = r$ is the solution we are seeking. If we let x_1 denote our initial approximation to r, then we can generally improve on this approximation by moving along the tangent line to $y = f(x)$ at x_1 until we meet the x-axis at a point x_2.....

The point-slope form of the tangent line to $y = f(x)$ at the initial approximation x_1 is $y - f(x) = f'(x_1)(x - x_1)$.

If $f'(x) \neq 0$ then this line is not parallel to the x-axis and consequently it crosses the x-axis at some point $(x_2, 0)$. Substituting the coordinates yields $x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$ ... In general if x_n is the nth approximation then the improved approximation x_{n+1} is given by $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$.

(Anton, 1995)

In Example 7.1, examples of simple **word-word** (WW) linkages are *initial approximation*, *point-slope* or *tangent line.* Examples of simple **symbol-symbol** (SS) linkages are, $f'(x) \neq 0$, $x = r$ and $y = f(x)$. Examples of **word-symbol** (WS) linkages include *x_n is the nth approximation* or *$x = r$ is the solution.* A simple **symbol-graphic** (SG) linkage is the connection between the illustration given in Example 7.1 and $y = f(x)$. A simple **word-graphic** (WG) linkage is the link between the graph and the word(s) '*crosses the x-axis*'. A **graphic-graphic** (GG) linkage may be the link between the graph $y = f(x)$ and the tangent line in the figure of Example 7.1.

Each simple linkage not only has the end components to consider but also the bridging between those components. The quality of simple linkages is identified in terms of the level of comprehension. The proposal is that each of the possible linkages between the words, symbols or graphics components can have a quality corresponding to any one of the six levels of comprehension. This does not mean that all linkages need to attain the highest comprehension level. In fact, the most appropriate level of comprehension may depend on the context in which the linkage is placed. To illustrate, let us return to Example 6.2 in Chapter 6.

The Wronskian for Differential Equations (Example 6.2 repeated)

The *Wronskian* of two differentiable functions y_1 and y_2 is denoted by $W(y_1, y_2)$ and is defined to be the function

$$W(y_1, y_2) = y_1 y_2' - y_1' y_2 = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

The value of $W(y_1, y_2)$ at a point x is denoted by $W(y_1, y_2)(x)$ or often more simply by $W(x)$. It can be proved that two solutions $y_1 = y_1(x)$ and $y_2 = y_2(x)$ are linearly dependent if and only if $W(x) = 0$ for all x. Equivalently, the functions are linearly independent if and only if $W(x) \neq 0$ for at least one value of x. Use this result to prove that the following solution of $\dfrac{d^2y}{dx^2} + p(x)\dfrac{dy}{dx} + q(x) = 0$ are linearly independent.

(a) $y_1 = e^{m_1 x}, y_2 = e^{m_2 x}$ $(m_1 \neq m_2)$

(b) $y_1 = e^{mx}, y_2 = xe^{mx}$

Now, consider possible simple SS linkages:

Example 7.2: Simple symbol-symbol linkages

1   1 or 2 or 3 or 4 or 5 or 6

$$W(y_1, y_2) \quad = \quad y_1 y_2' - y_1' y_2 \quad = \quad \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

2   2

$y_1 = e^{m_1 x}, y_2 = e^{m_2 x}$ $(m_1 \neq m_2)$ Also y_1 and y_1'

4   3

The numbers in the diagram represent the different levels of comprehension for each linkage.

1 : Simple labelling or definition SS linkage. Meaning is not present. (*Literal*)

2: Recognition SS linkage. For example, y_1 in the question links with y_1 in the definition as one to be substituted into the other. No meaning needs to be present. (*Organisational*).

3: A process is recognised. If this process has no meaning could be *organisational* (2), otherwise it represents *inferential* comprehension.

4: Apprehending conditions. For example, $m_1 \neq m_2$ implies a situation of two discrete roots to a quadratic equation in m as opposed to complex or the same roots. (*Evaluative*).

5: Apprehending a whole phenomenon. (*Appreciative*)

6: Apprehending the whole phenomenon, value judgements and use of the phenomenon. (*Integrative*)

An example of one simple SS linkage that illustrates a range of comprehension is

1 or 2 or 3 or 4 or 5 or 6

$$y_1y_2' - y_1'y_2 = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

If the student recognises the symbols y_1, y_2, y_1', y_2' on each side of the equation and the symbol and position of elements for the determinant, it is level (**1**). If they can also recognise the formula for finding the determinant $y_1y_2' - y_1'y_2$ and know how to substitute one value into another, it is level (**2**). For each of these two situations there is no meaning involved.

If the student recognises the link as a <u>process</u> for finding a determinant, for example, 'multiplying leading minus multiplying trailing', it is level (**3**). If they understand the conditions on a determinant and its properties in context it is level (**4**). To obtain a level (**5**), the student would need to link determinants with linearly dependent or independent equations. If the student has integrated a high level of comprehension and has made a value judgement on the linkage and can explain why the formula is appropriate for differential equations, it is level (**6**).

This idea of context determining the level of comprehension can also be applied to the other types of simple linkages and a mixture of those linkages. To illustrate this, let us take a student's solution to the Wronskian question in Example 6.2 as seen in Examples 7.3 and 7.4. As the student is writing the solution they are periodically going back to the passage and forming new links with sections of that text. These linkages are expressed by the student in written form. The student is also reading and comprehending their own writing.

Example 7.3: A student's solution to the Wronskian question

$y_1 = e^{m_1 x}, y_2 = e^{m_2 x}$ $\qquad y_1' = m_1 e^{m_1 x}$ $\quad y_2' = m_2 e^{m_2 x}$

$$
\begin{aligned}
y_1 y_2' - y_1' y_2 &= e^{m_1 x} m_2 e^{m_2 x} - m_1 e^{m_1 x} e^{m_2 x} \\
&= m_2 e^{x(m_1+m_2)} - m_1 e^{x(m_1+m_2)} \\
&= e^{x(m_1+m_2)}(m_2 - m_1) \quad \text{but } e^{x(m_1+m_2)} \neq 0
\end{aligned}
$$

$m_1 - m_2 = 0 \quad \Rightarrow \quad m_1 = m_2 \qquad$ But $m_1 \neq m_2 \quad \Rightarrow \quad m_1 - m_2 \neq 0$

$\therefore W(x) \neq 0$, linear independent

$\therefore y_1 = e^{m_1 x}$ and $y_2 = e^{m_2 x}$ are linearly independent.

Example 7.4: A student's solution showing linkages

1 (SS) $\qquad$ **Any (SS)**

$$\left[\; W(y_1, y_2) \quad = \quad y_1 y_2' - y_1' y_2 \quad = \quad \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} \;\right]$$

2 (SS)

$y_1 = e^{m_1 x}, y_2 = e^{m_2 x}$

3 (SS)

$y_1' = m_1 e^{m_1 x}$ $\quad y_2' = m_2 e^{m_2 x}$

2 (SS)

$y_1 y_2' - y_1' y_2 = e^{m_1 x} m_2 e^{m_2 x} - m_1 e^{m_1 x} e^{m_2 x}$

2 (SS)

$= m_2 e^{x(m_1+m_2)} - m_1 e^{x(m_1+m_2)}$

2 (SS)

$= e^{x(m_1+m_2)}(m_2 - m_1)$

3 (SS)

but $e^{x(m_1+m_2)} \neq 0$ $\quad$ (**4 (SS) or 4 (SG)**; Knowledge about $e^x \neq 0$ imported, can be symbolic or graphic.)

3 (SS) or 4 (SS)

$m_1 - m_2 = 0 \quad \Rightarrow \quad m_1 = m_2$

2 (SS) $\qquad$ **4 (SS)**

But $m_1 \neq m_2 \quad \Rightarrow \quad m_1 - m_2 \neq 0$

2 (SS)

[Therefore $W(x) \neq 0$, linear independent]

4 (WS) or 5 (WS)

[This part of the solution also implies **1 (WS)** for
$W(x) \neq 0$ - linear independent and
$W(x) = 0$ - linear dependent.] $\qquad$ **4 (WS) or 4 (SS) or 5 (WS)**

$\therefore y_1 = e^{m_1 x}$ and $y_2 = e^{m_2 x}$ are linearly independent.

5 (WS)

Example 7.3 is a transcription of the student's solution and Example 7.4 is the same solution with possible linkages inserted with the author's estimate of the level of comprehension required for each linkage.

Example 7.4 displays predominantly **symbol-symbol** linkages. **Word-symbol** linkages do not play a part until the end of the solution, especially in the interpretation of the symbolic answer. This example illustrates simple linkages within a particular context and is used to highlight the variety of levels and types of linkages common in both reading comprehension and comprehension when writing solutions. The same ideas can be extended to other contexts whether they involve predominantly symbols, narrative or graphics or a mixture of these three components.

Therefore for layer 2 of mathematical text, although a simple linkage between two components may have different possible levels of comprehension, the most appropriate comprehension of dominant or critical linkages may depend on the context in which the linkage exists within the text. It is the context of each linkage that promotes or mediates the depth of comprehension of each linkage. From the studies in the previous chapter, it appears that weaker students only manage to comprehend links at levels **1** or **2,** irrespective of whether this is the appropriate level for the context of the linkage. This reinforces a surface approach to reading mathematical text. The fact that calculations often require only levels 1 or 2 comprehension in each link may explain why those same students weak in comprehending concepts can still perform some calculations that do not require more than a low level for comprehending linkages.

7.3.3 Layer 3 - Section or paragraph comprehension

Within any section of mathematical text, not only are the linkages between components numerous, but as illustrated in the previous section, students are unlikely to have the same level of comprehension of each linkage. How then does the student gain overall comprehension of a passage of text?

It must be assumed that within a complicated network of linkages there are dominant (or critical) links that contribute to overall comprehension. Therefore layers 3 and 4 are different from layers 1 and 2 in terms of complexity. To illustrate, the following is a theorem taken from 'Calculus' (Anton, 1995) and was used in Chapter 6. This time, the theorem is revisited for analysis in terms of our model.

10.2.1 THEOREM **(L'Hopital's Rule for Form 0/0).** *Let* lim *stand for one of the limits* $\lim\limits_{x\to a}$, $\lim\limits_{x\to a^+}$, $\lim\limits_{x\to a^-}$, $\lim\limits_{x\to +\infty}$, or $\lim\limits_{x\to -\infty}$ *and suppose that* $\lim f(x) = 0$ *and* $\lim g(x) = 0$. *If* $\lim\ [f'(x)/g'(x)]$ *has a finite value* L, *or if this limit is* $+\infty$ or $-\infty$, *then* $\lim \dfrac{f(x)}{g(x)} = \lim \dfrac{f'(x)}{g'(x)}$.

The main idea from the theorem lies in the dominant SS linkages between $\lim f(x) = 0$, $\lim g(x) = 0$ and $\lim \frac{f(x)}{g(x)}$ to show that the theorem relates to the form 0/0; between $\lim \frac{f(x)}{g(x)} = \lim \frac{f'(x)}{g'(x)}$ to show the equality of the two limits; and the change from $\frac{f(x)}{g(x)}$ to $\frac{f'(x)}{g'(x)}$.

<u>Literal comprehension</u> is plausible when the student can recognise the symbols but give an explanation of the theorem as a literal translation from symbols into word form. For example, $\lim \frac{f(x)}{g(x)} = \lim \frac{f'(x)}{g'(x)}$ was explained as:

> *The limit of one function divided by another is equal to the limit of the derivative of that function divided by the derivative of the other.*

The student read the symbolic expression literally and rewrote the expression into narrative form with a literal translation. The meaning of the theorem or expression is not conveyed. In the above quote, even though the overall comprehension is literal, the student displayed a slightly higher comprehension of a basic linkage by recognising $f'(x)$ as the derivative of $f(x)$. Another literal example for layer 3 is:

> *The limit of the function f (x) over g (x) equals that of limit f dash over g dash.*

In this case, the student recognised $f(x)$ and $g(x)$ as functions, but perhaps has not recognised $f'(x)$ as a derivative of the function $f(x)$.

These two examples indicate an overall literal level of comprehension and demonstrate how the comprehension of basic components can differ but without affecting this overall comprehension level.

<u>Organisational comprehension</u> of layer 3 occurs when the main ideas and some of the more obvious details are recognised.

> *If lim f(x)=0 and lim g(x)=0, then to find the limit of the question, derive both the f(x) and g(x) and sub the limit into its derivative. This will give the limit.*

The student has recognised the dominant information or process but did not demonstrate understanding of the question. The student is describing the theorem as a process that will achieve the required end product.

Inferential comprehension of layer 3 occurs when the student comprehends the dominant ideas and obvious details. The difference between inferential and organisational comprehension lies in the rudimentary understanding of the main ideas. Inferential comprehension shows some understanding.

> *If the function's* $(\frac{f}{g})$ *limit equals* $\frac{0}{0}$*, then L'Hopital's rule says the derivative's* $(\frac{f'}{g'})$ *limit can be taken as the answer to the function's limit.*

This student explained the theorem from the perspective of L'Hopital's Rule. Not only has s/he connected $\frac{f}{g}$ with $\frac{0}{0}$ but they have made the leap in meaning by inferring that the limit of $\frac{f}{g}$ can be determined (under certain conditions) by finding the limit of $\frac{f'}{g'}$. Both limits are equivalent. Even though the student has missed out other details, this rudimentary meaning displays an overall inferential comprehension of the theorem.

Evaluative comprehension is conveyed when the student can not only adequately comprehend the main point and many of the details, but they also connect to previous knowledge. They show more than some rudimentary understanding of the topic.

> *If a function has both numerator and denominator approaching zero it is difficult to work out the limit. L'Hopital's rule says that if we differentiate the top and bottom of the equation and take that limit, if it comes to a finite number or* $-\infty$ *or* $+\infty$*, then this answer will be the same as the limit of the original function (but only if the original function's limit is* $\frac{0}{0}$*, and both f (x) and g(x) must be differentiable).*

The student can evaluate why the theorem may be necessary, how the process works, what the end result was, and link previous knowledge relating to conditions for differentiability of functions.

In order to achieve *appreciative comprehension*, extra knowledge beyond the information given in the theorem at the beginning of this section is needed. The student would see L'Hopital's Rule as a phenomenon. This may include the proof of why it is acceptable that $\lim\frac{f(x)}{g(x)} = \lim\frac{f'(x)}{g'(x)}$ in addition to linking with previous knowledge such as the extended mean-value theorem. A further requirement may be the comprehension of limits and the conditions under which L'Hopital's Rule would work (such as for $\lim\limits_{x \to a} \frac{\infty}{\infty}$). Further evidence of appreciative comprehension could be to question why the rule applies to $\lim\limits_{x \to a}$, $\lim\limits_{x \to a^+}$, $\lim\limits_{x \to a^-}$, $\lim\limits_{x \to +\infty}$, or $\lim\limits_{x \to -\infty}$ equally.

Finally, for *integrative comprehension* the student not only fully comprehends the theorem, but they can abstractly extend, use, apply and form an opinion about its value. Such a comprehension level may be difficult to obtain, especially with only a section of text, and is likely to occur only after a high level of comprehension of layer 4 is attained.

7.3.4 Layer 4 - Topic comprehension

Layer 4 is similar to layer 3 in that dominant or critical linkages play a major part in overall comprehension. However, comprehension of layer 4 not only includes comprehension of layers 1, 2, and 3 but also how and to what degree students link together the various major sections or components of a text. A typical mathematical text has a set order of sections: the introduction, definition, theorem, summary or brief explanation and worked examples. In comprehending large pieces of mathematical text, some sections can contribute more to the overall knowledge of the concept(s) displayed in the text than others. This was acknowledged by many of the students in the extract reading experiment in Chapter 6, Section 6B (& Appendix E). Therefore comprehension may depend on whether the students concentrate on one section and skim over the others, or whether students concentrate on several sections. The contribution of each section and the quality of links developed between sections must affect the overall comprehension of a topic. Each section contributes something to the overall comprehension. A student missing the introduction may not comprehend why the concept is needed; missing the definition ignores what the parts of the concept represent; missing the theorem ignores a central abstract summary of the concept; missing the explanation ignores how the concept is developed; and missing the worked examples ignores an illustrated application of the concept.

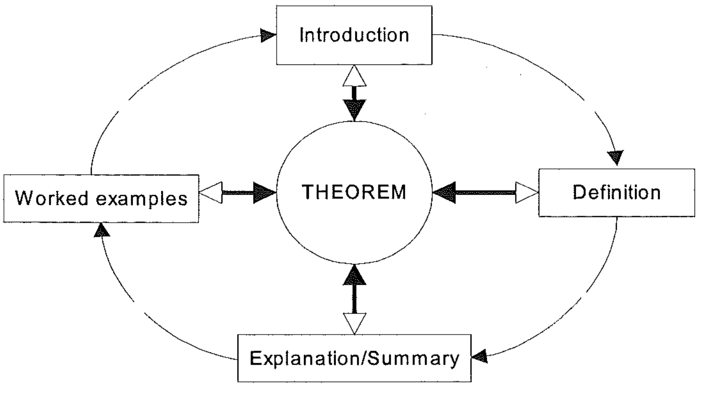


Figure 7.2: Relationship between sections of text

In a topic, the theorem may be the central component to comprehend (Figure 7.2). This was acknowledged as one of the hardest parts to comprehend as illustrated in Chapter 6, Section 6B. In a mathematical topic, the introduction leads into the theorem, the definition defines the concepts in the theorem, the explanation extends the concepts in the theorem in more detail and the worked examples are numerical illustrations of the theorem. Therefore concentrating on the theorem and linking the other sections to the theorem should give greater opportunity for higher comprehension. Between 44% and 51% of the students doing the reading extract experiment did this (Appendix E). The outer unidirectional cyclic ring of arrows in Figure 7.2 represents the text author's sequence of presenting topics, with the theorem being incorporated predominantly at either the definition or explanation stage. Sometimes the theorem is omitted altogether from a text. The dominant linkages are represented by the darker arrows. The reversible arrows in Figure 7.2 depict the interrelationship between sections.

A concentration on the worked examples as the central component would need further effort to generalise the concept into abstract form and comprehend counter-examples. So, although all sections are relevant for comprehension some sections may contribute more than others and this may be a justification for some students skipping the odd section as illustrated on pages 6-21 to 6-23.

7.4 Summary

Of the four layers outlined in Figure 7.1, layer 4 most closely resembles Dechant's levels of comprehension since Dechant referred to the text as a whole. However, layer 4 is the linking between pieces of text and each piece itself requires comprehension. Likewise within each piece of text both the basic components and linkages between those components require comprehension. Therefore we can say that layer 4 incorporates, and is affected by, layers 1, 2 and 3. Similarly, layer 3 incorporates and is affected by layers 1 and 2. Layer 2 is influenced by comprehension of basic components (layer 1).

Logically it should follow that a lower comprehension level obtained for layer 4 would be heavily influenced by the lower layers, in particular the basic components and simple linkages within sections of text. We can see this with Dechant's levels of comprehending. For example, literal comprehension of a topic is the adequate comprehension of individual words, symbols or simple linkages in explicit or literal form. Concentration on understanding worked examples or just following the structure of the sections in the text illustrates examples of organisational comprehension, where meaning is still not involved. Here, the layer 2 dominant linkages only require relatively low comprehension levels. If the student comprehends the main ideas and some of the details or shows some rudimentary understanding of more than one section they have achieved inferential comprehension. If the student shows greater understanding of the text by incorporating the conditions and counterexamples and links the topic to some of their background knowledge, they have achieved evaluative comprehension. When the student sees the concept or topic as a whole phenomenon they have reached appreciative comprehension. When they can use this topic in other contexts and place value judgements on the topic, they have achieved integrative comprehension.

Chapter 8 Comprehension with self-learning technology

New software is being created and marketed as self-study for undergraduate mathematics students. In this chapter, some of these alternative resources are explored in terms of mathematical comprehension at the undergraduate level.

8.1 Introduction

With the advancement of technology and the introduction of larger portable storage systems, first year textbook authors have begun to offer computer-based (CD ROM) versions of their text, first as a supplement and later as an alternative for textbooks such as Larson, Hostetler, & Edwards, (1995. Although hard-copy text print is still a cheaper and more accessible option than these other self-study learning resources, this recent introduction of mathematical software packages has the potential to change the direction of independent tertiary reading and learning.

The presentation of content on the CD ROM is still structured by the author but it aims to give the learner easier access to concepts, examples, exercises and applications. Animations or video clips are included to illustrate concepts. A more recent development, relevant to this study, is the design of computer-based learning programs that cover specific mathematical content at the university level (for example, (Quinney, Harding, & Intellipro, 1996; Monash University, 1997). These programs are not simulations but rather interactive software specifically designed for self-learning purposes.

The survey in Chapter 6 pointed to under-use of hard-copy text reading material. If the students are reluctant to use textbooks and printed text for independent study, would the same material presented through educational technology, such as the software packages that accompany textbooks or are independent courseware in their own right, improve students' willingness to read and understand mathematics? How do we know whether the software being presented as substitutes for textbooks improves student reading and understanding of basic mathematical concepts?

If there are only small positive gains in computer-assisted and computer-based technology at the tertiary level, as proposed by Tjaden and Martin (1995), why consider using computer packages instead of printed text? Why compare the packages? First, the self-study software packages available today are recent developments that have not yet been evaluated in terms of student learning. Second, many of the positive attributes associated with computer-assisted or computer-based learning are supposed to be incorporated into these software packages designed for self-study. This should make the packages compatible with the skills required for mathematical comprehension of expository text. The self-study software packages available today may have additional gains over printed text that includes the non-sequential approach of content allowing for more learner choice, the combination of graphics, animation and text, and a user-friendly interface that requires little or no training. A further advantage is the novelty of using a new package, although there is some evidence to indicate that any such enthusiasm is likely to be a temporary phenomenon and not present in the long-term (Lawson, 1995).

This chapter explores mathematical comprehension using some of the current software packages. The four learning resources are divided into two groups:

- Non-interactive learning resources
  - ◊ Hard-copy text (textbooks)
  - ◊ Software that does not involve direct student input

- Interactive learning resources
  - ◊ Text-based software
  - ◊ Multimedia software

In Section 8A, the technology is assessed as a comparison to comprehension of hard-copy text. In Section 8B one of the software packages is used within a tutorial situation as an exploration of longer-term use and its effect on the development of comprehension over the first year of undergraduate study.

Section 8A

8.2 Technology versus textbook

The aim of this chapter is to compare the effectiveness of several possible types of learning resources for self-study within first year undergraduate mathematics. A standard first year university textbook is compared with three different software packages taken from each of the groups mentioned on the previous page. Each of these four learning resources is assessed in terms of the content (that is concepts, procedures and applications) comprehended by a trial group of students and in terms of learning behaviour. Can any of these software packages be a viable alternative to the textbook for self-study purposes?

8.2.1 The students

The study was conducted in 1997 and was repeated in 1998. A total of 47 students took part, N=14 in 1997 and N=33 in 1998. All participants were volunteers from a first year mainstream undergraduate mathematics course. Approximately 90% of the students enrolled in the mainstream course were aged 18 to 20 years, one third were of Asian descent and 27% were female. On ability, 68% rated themselves as "average" and 24% as "above average". The textbook for the mainstream course was regarded positively by 49% of the class and negatively by 36%. The sample of students taking part in this study closely reflected these figures.

8.2.2 Method

For both years the tasks were conducted during a three week university lecture break in a small tutorial room with only the student and author present. Students took part in one session and used one of the four learning resources. Approximately 12 students were allocated to each learning resource. Students performed a given task either alone or in pairs. Sessions lasted from 70 to 100 minutes in total and began with 5 to 10 minutes for task instruction, 5 minutes for a pre-test, 34 to 52 minutes on the allotted task, 10 to 20 minutes on a post-test and 10 to 15 minutes on taped interviews. The student decided when the task was completed. Taped interviews supported post-test answers and gave the student the opportunity to discuss both the task and the learning resource. Notes taken by students during the sessions were included in the analysis and discussion between pairs was taped. In addition, in 1997 the author used a minute by minute timeline to record a student's reading behaviour including time spent backtracking, skipping text or re-reading the same passage.

It was difficult to find the same coverage and depth in one topic handled equally by each of the four different learning resources. The topic eventually chosen was *homogeneous second order differential equations* and (for both years) this study was conducted prior to the topic being officially covered in lectures. Although first order separable differential equations are studied in the final year of secondary school, second order differential equations were expected to be new to the students. Prior knowledge was evaluated with a pre-test that asked students about some basic ideas on both general and second order differential equations. Lack of student prior knowledge in the topic resulted in many students being unwilling to write their answers on their pre-test sheet. Therefore pre-test answers were reaffirmed with a taped interview prior to the task. The post-test questions were the same as the pre-test. Efforts were made to eliminate external distractions and provide the students with a quiet comfortable environment.

One of the tasks required the students (N=12) to read, either alone or in pairs, a pre-selected section from their course textbook, labelled in this article as "Textbook" (Anton, 1995). Students were informed that they need not confine themselves to this section if they felt further information could be obtained from another section of the book.

Students doing another task (N=12) used a computer to read an abridged textbook that had been placed on a CD ROM (Larson et al., 1995). This program, labelled for the study as "CD-text", contained brief accounts of the concepts with examples and a variety of exercises with easily accessible solutions. There were some rudimentary animations and graphic clips.

In another task the students (N=11) used a text-based interactive software program 'Epsilon' (Monash University, 1997) that contained graphics, some animation and a basic summary of concepts accompanied by a few exercises. 'Epsilon' was released in early 1998 and was not available for the 1997 study. The final task being compared required the students (N=12) to use a multimedia-based interactive software package on a CD ROM, labelled for this study as "Multimedia" (Quinney et al., 1996). Apart from some text reading, the multimedia package contained a video clip, considerable animation and the opportunity to alter variables that resulted in immediate graphical changes.

8.2.3 Results

The results are divided into three categories. The first is a timeline account of the reading behaviour recorded during each session. This was done in 1997 and therefore covers three of the four learning resources. The second is a comparison between pre-test and post-test results for both years, supplemented by interviews. The third explores the students' opinions of the learning resources they used.

Expository reading behaviour

In every session in 1997 the author recorded minute by minute student reading behaviour. This recording was to determine the different reading elements and the time students spent on each. For example, reading a passage without re-reading any text, local re-reading of the same small passage of text (indicated by the amount of time spent on the same passage), skipping large passages of text, re-reading over text previously covered (labelled 'backtracking') and doing exercises or watching a video. Also recorded was the average amount of time students spent using a pen and paper to take notes or do exercises. The results are outlined in Table 8.1.

Table 8.1: ***Average percent of time spent on behavioural elements in reading an expository text.***

	Textbook		CD-text*	Multimedia	
% (average)	individual	pairs	individual	individual	pairs
forward reading	60	40	46	44	37
re-reading the same text	5.5	43	16	23	26
skipping text	6	0	7	12	9
backtracking	21.5	7.5	15	7	5
exercises	7	9.5	13	0	0
video	0	0	0	12	23
pen and paper	46	43.5	35	14.3	0

* In 1997 pairs were not used on this task. The program 'Epsilon' was only available for the 1998 study.

It appears that students use similar essential elements for expository text reading independent of the learning resource used. Differences appeared to be in the emphasis. Students using hard-copy text on their own spent comparatively more time forward reading, less time re-reading the same passage and comparatively the highest proportion of time backtracking by flipping back to previous pages. In contrast, student pairs using the same hard-copy text spent more time discussing the same passage (that is, comprising most of the 43% entry for 're-reading the same text') and would not advance to the next passage until they felt they had achieved a rudimentary understanding. These Textbook task students used pen and paper more often than the students using the computer packages.

The multimedia package as a learning resource did not encourage the use of pen and paper, backtracking or discussion, but rather encouraged comparatively more skipping through text without intensive reading and more time spent on video clips and interactive graphics. For the students using Multimedia up to 23% of the time was spent watching the video and animation clips. Students originally tried to understand the text but eventually read at a quicker speed and later acknowledged their limited comprehension of the text.

In comparison to both the Multimedia and Textbook, students using the CD-text occupied a middle ground time-wise and displayed the highest frequency for using the hypertext buttons to change between exercises and concepts.

Detailed timelines were not recorded for the 1998 sessions but observations were taken. The 1998 student expository reading behaviour was similar to the 1997 results in Table 8.1. Students using 'Epsilon' and pairs of students using the CD-text exhibited similar behaviour to the 1997 individuals using the CD-text.

Pre-test and post-test results

Although most of the questions given in the pre-test and post-test were conceptual rather than procedural, the concept questions were answered quickly while the procedural question took much longer. The questions were:

Q1 What is a differential equation?
Q2 What is meant by a first and second order differential equation?
Q3 What is an auxiliary (or characteristic) equation?
Q4 Explain the difference between a particular solution and a general solution.
Q5 Give an example of a physical application modelled by a differential equation.
Q6 Solve $y'' + 7y' + 12y = 0$ where $y(0) = 0$ and $y'(0) = 2$.

The answers to the pre-test and post-test were collated from written scripts and taped interviews. The *accuracy and depth* of student knowledge to each of seven questions was placed on a scale from 1 to 5 as follows:

0	1	2	3	4	5
did not attempt	attempt but incorrect	a small amount	about half	more than half	most or all

It was not enough for students to give a definition. They were also required to indicate the depth of their understanding. This was obtained from the interviews in conjunction with post-test written answers. In Table 8.2 the first entry value is the average pre-test score and the final entry after the arrow is the average post-test score. Table 8.2 therefore records the relative progress attributable to each learning resource. Using *t*-tests, all four learning resources showed a significant improvement from the pre-test to post-test scores (see Appendix F).

Table 8.2: Results of average pre-test and post-test scores for each learning resource

	Q1 concept	Q2 concept	Q3 concept	Q4 concept	Q5 application	Q6 procedure	overall average
Textbook	0.7→3.3	0.8→3.9	0→2.3	0.3→3.5	0.7→1.2	0.7→4.4	0.5→3.1
CD-text	0.5→4.2	0.5→4.6	0→4.2	0.5→4.3	0.5→4	0.5→4.2	0.4→4.2
'Epsilon' (Monash)	0.5→3	0.2→4.1	0→4	0.1→4.2	0.4→4.2	0.4→4.5	0.3→4
Multimedia	0.3→1.8	0.6→2.8	0.1→1.5	0.6→2.5	0.3→3.5	0.2→1.1	0.4→2.2

Students using the CD-text and 'Epsilon' were the most accurate on the post-test scores for knowledge of both general and specific concepts, describing a physical application and procedural accuracy (Table 8.2). These students appeared to gain the most comprehensive all round knowledge of the topic.

In the pre-test the students using the textbook often referred to velocity for their application, but after completing the task they thought that a physical application meant a procedural calculation. These students using the textbook scored well on general concepts and procedural accuracy but poorly on questions relating to modelling of applications and second order differential equations. In contrast, students using the multimedia package scored poorly in all areas except describing the physical application presented in the video and animation clips where they scored the best.

For our sample a ranking of the learning resources in terms of relative progress was that the CD-text and 'Epsilon' were equally effective, followed by the Textbook and finally the Multimedia (Figure 8.1). This applied to both the individual and pair sharing of each resource (see Appendix F).

Although Figure 8.1 shows that pairs scored higher than individuals for three of the resources, these differences were not necessarily significant using *t*-tests. Between the first four items in Figure 8.1, there was no significant difference. However, CD-text and 'Epsilon' pairs scored significantly better than the students using Textbook and Multimedia. The t-test scores are given in Appendix F.

Figure 8.1: Ranking of learning resources based on post-test scores for individuals and pairs

	CD-text pairs	highest post-test scores
significantly better	Epsilon pairs	
	CD-text individual	
	Epsilon individual	
	Textbook pairs	
	Textbook individual	
	Multimedia individual	
	Multimedia pairs	lowest post-test scores

Although Textbook pairs gave a higher score than Textbook individual, the pairs did not score significantly higher than the individuals. In contrast, the Multimedia individual score was significantly higher than the Multimedia pair scores. The ranking in Figure 8.1 for Textbook pairs was below 'Epsilon' and CD-text, although the t-tests indicated that the Textbook pairs were not significantly lower than individuals working on 'Epsilon' and CD-text.

Opinions of learning resources

At the end of each session, students were interviewed using open-ended questions about the content and what they liked or disliked about the learning resource they used. Students using the Textbook task liked the exercises but not the textbook's approach. They felt that many parts were not explained clearly. For example, they could not understand where the trial solution of $y = e^{mx}$ came from. One student also commented that the exercise questions at the back of the section did not relate to the text. Those using the Textbook task individually felt they had not really understood the topic by the end of the session. However the students who worked in pairs with the Textbook task commented on the ease of learning achieved by working with someone else.

> *...Stopped you from just going on (through the work)...I often did not understand how you got it, but T knew how to get it...and then it would be really easy.*
> *(In fact, this student helped the other partner just as often.)*

The pair-sharing students also felt they got through the topic faster even though student pairs took longer to complete the task. These students also commented that they would have given up sooner if they had not been working with another student.

The students using the CD-text liked the non-sequential approach and felt they could find information more easily with this resource than with their textbook.

> *You can flip to wherever you like, you don't have to read through pages of stuff.*

Their only concern was access to a computer and the fact that a computer takes up more space than a textbook. The students also commented on the ease of access to relevant exercises and the fact that detailed solutions were available as opposed to an answer located at the end of a textbook.

The ability to '*flip to wherever one likes*' also appealed to the students using Epsilon. However this program was criticised for its lack of detail and limited worked examples. 'Epsilon' did have some worked examples, but the students felt these jumped several levels of difficulty.

The younger students using the multimedia liked the video and animation clips but the more mature students found them distracting and irritating. All students felt they understood very little about the topic because "clear-cut definitions of the basics was missing". The graphs that could change when the student altered a variable meant little to the younger students who did not appear to try to understand the underlying message. The interactive graphics caused confusion for the more mature students who made an effort to understand the changes. All students doing this task stated that they traditionally used a textbook to do exercises and therefore missed not being able to do the same with the multimedia resource. The students on this task admitted they could not follow most of the topic presentation. The apparently non-sequential approach of the package did not appeal to the students even though the students themselves dictated when their task was completed:

> *It's difficult to know what to go through...need more time...need more examples.*

One of the more mature students commented that the multimedia approach was not useful as a first up learning option as he would prefer a lecturing situation first. He felt the package could be useful for finishing off a topic.

Section 8B

8.3 CD-text in tutorials

In this section, one of the more successful software packages from Section 8A was introduced into a first year tutorial environment. Weekly one-hour tutorials were accompanied by four formal hours of lectures. Each tutorial contained approximately 15 students. In 1998 there were approximately 270 students in each lecture stream and 15 students in each tutorial. The purpose of general first year tutorials is to allow students to practice relevant exercises, ask questions and obtain help. As mentioned earlier, before attending each tutorial students were expected to bring with them their solutions to a set number of preparation questions selected from the textbook. More difficult questions were set in the tutorials themselves.

In 1998, four tutorials were set aside in order to compare the effect of using a self-learning software program within tutorials. Two of the tutorials took place in a computer laboratory where students had their own computer and CD. The computer program selected was the CD-text assessed in Section 8A. The results of these two tutorials were combined and labelled for this study as *Computer Group*. The other two tutorials were combined as a *Control Group* and were conducted in a traditional classroom environment.

8.3.1 The tutorials

Both the *Computer Group* and the *Control Group* attended the same four hours of lectures each week. Both the groups received the same preparation exercises from the textbook prior to tutorials. However, the *Computer Group* did not have access to their textbook during tutorials, but were allocated questions in which the solutions were within easy access with the click of a mouse. The *Control Group* had access to their solutions at the end of the week.

8.3.2 Selection of students for the study

Of the 580 original students in the course, 196 volunteered to be involved in the computer tutorials. Those students eventually chosen for the two computer tutorials were randomly selected from the students preferring popular time slots. The students for the *Control Group* did not necessarily choose to take part in the study and allocation was dependent on student preferences for tutorial times. Overall, a total of 30 students were allocated to the *Computer Group* and 30 to the *Control Group*. The demographics for each group were:

Table 8.3: Demographics with frequency of students

	Computer Group	Control Group
Mature Male (>25 years)	7	6
Mature Female (>25 years)	2	2
Young Male	11	12
Young Female	10	11
Total	30	30

Some precautions were used to ensure a reasonably homogeneous population and consistent treatment. Students obtained between 50% and 75% in the final secondary school mathematics examination, thus eliminating extreme abilities. Instructions and explanations to the class were exactly the same for both the *Computer* and *Control Groups* in both presentation and content. For example, most students experienced difficulty with the second fundamental theorem of calculus and maintained that they could not understand the explanations in either the lectures or the textbook. For that week the groups received the same help, in this case a class/teacher build-up of concepts in the first fifteen minutes of the tutorial hour.

8.3.3 Computer tutorials

At the end of the 13th week the Computer Group students evaluated the computer tutorials. The majority of students, when given the option, requested an alternation of computer and traditional tutorials and this was done for the following six weeks. By this time the results of the first class test were available. After further evaluation by the students the final six weeks of the academic year changed from computer to traditional tutorials. During these final six academic weeks the students were encouraged to use the CD-text for self-study outside formal contact hours.

There was some student attrition for both groups. By the end of the 13th week, there were 28 of the original *Computer Group* (N=30) students and 27 of the *Control Group* (N=30). By the end of the 25th academic week, there were 20 of the original 30 students left in the *Computer Group* and 23 of the 30 original students left in the *Control Group*. Of the ten *Computer Group* students who eventually dropped out, two preferred traditional tutorials, three were finding the course too difficult and changed courses and five students decided that the 5% allocated to tutorials was not worth their effort and so they chose not to attend. Of the seven students in the *Control Group*, three students dropped out of the course after their first test marks showed a failure and four students were 'too busy' to attend. Two of these latter four students returned in the last six weeks of the year after they failed the second class test.

8.3.4 Results

Observations

Use of the CD ROM program: The *Computer Group* was guided to the sections and exercises relevant to each tutorial, but also had easy access to other aspects of the topics. All the students in the *Computer Group* began the year by going directly to the exercises and using pen and paper. By the second week seven of the nine mature students spent half their tutorial time reading through the concepts and taking notes. These students gave favourable comments about the clarity of the explanations on the CD-text. However, when the students encountered the topic on "limits" most of the students spent a lot of the computer tutorial going through the worked examples rather than the exercises. By the end of the 12th week, all nine mature students and five of the more able younger students were spending approximately 40% of their time on the concepts, 40% of their time on worked examples and 20% of their time on exercises. Most of the younger students (except for the more able students) concentrated on either the worked solutions or the exercises. A distinctive feature of all the younger students was the decreasing use of pen and paper. These students would spend much of their tutorial time looking at the questions for a few minutes, looking for parts they recognised, determining a vague outline of what to do (all in their heads), accessing the solution and studying the details.

The control group

The *Control Group* students were given sets of exercises from their textbook. Although the students had their textbook on hand, throughout the year they consistently confined themselves to the set exercises. Many of these students said that they only looked at the rest of the textbook in order to find a similar example to the exercise they were working on. As a result, any requests for help from the tutor involved some aspect of the calculations rather than a concept query. Throughout the year there was little effort invested in understanding the concepts during tutorials.

An overall impression is that the computer tutorials appeared to succeed in broadening the students' approach beyond the set exercises.

Test Results

The entire class of 580 students sat two one and a half hour tests during the academic year. Some parts of the tests were procedural: For example

Find the derivative of the following expressions. You need not simplify your answers:

$$\frac{d}{dx}\frac{\sqrt{x}\sin(x)}{x}, x > 0$$

Other questions involved a marginally higher level of comprehension. For example:

Let $f(x) = x^2 \ln(x)$ for $x > 0$. Find all the regions of increase and decrease and all local extrema of f. Determine the behaviour of $f(x)$ as $x \to \infty$ and as $x \to 0^+$. Also determine the behaviour of $f'(x)$ as $x \to 0^+$.

The average for Test 1 was 37.9 (s.d.=11.13) and for Test 2 was 29.4 (s.d.=11.38). Both the *Control* and *Computer groups* were not significantly different from the class scores and illustrated in Table 8.4.

Table 8.4: Mean, standard deviation and z-scores for Tests 1 and 2

	Test 1	Test 2
Class (N = 580)	37.9 (s.d. = 11.13)	24.9 (s.d. = 11.38)
Control Group (N = 30)	37.2 (s.d. = 13.31)	29.4 (s.d. = 14.12)
Computer Group (N = 30)	37.8 (s.d. = 16.3)	26.3 (s.d. = 15.8)
Class versus Control Group (t-score)	0.11 (not significant)	1.34 (not significant)
Class versus Computer Group (t-score)	0.03 (not significant)	0.48 (not significant)

Observed advantages of the computer tutorials over traditional tutorials

Overall, the *Computer Group* students became more independent, enthusiastic and less reliant on instructions and help than the *Control Group.* For example, with the *Computer Group* several students volunteered to collect the CD's and topic sheets prior to the tutorial and at least half the class were already into the work five minutes before official start time. Once into the tutorial, students in the *Computer Group* asked fewer questions and any that were asked usually dealt with concepts and why an operation took place. The immediate availability of detailed solutions was favourably commented on and well used by the students. In contrast, the *Control Group* waited to be told to start, many students were a few minutes late and during tutorials their numerous questions related to how to begin the questions. Not surprisingly, the *Computer Group* students became faster at going through the same tutorial work than the *Control Group.* One student from the *Computer Group* once went to a traditional tutorial (taken by another tutor) and commented:

> *I couldn't believe how slow the others were going through the (traditional) tutorial questions...they seemed to take forever and only went through one of the questions.*

Therefore access to detailed solutions appeared to prompt a different type of question. It should also be noted that students from both groups still had difficulty deciding where to begin if they were given new questions. It also needs to be noted that the *Computer Group* came to tutorials more prepared than the *Control Group.* However when the *Computer Group* changed to alternating between a computer and traditional tutorial, preparation for tutorials dropped off and by the time the group became a traditional tutorial both the *Computer* and *Control Groups* were attending tutorials with very little preparation. One influencing factor for both groups later in the year could be the increase in workload, not only in mathematics, but in other subjects as well.

Observed disadvantages of the computer tutorials

Students using the CD-text to do exercises tended to concentrate only on those questions that had a solution immediately available. There was a reluctance to try the exercises that did not have detailed solutions available at the click of a mouse. The computer tutorials, as used in the study, also gave a false impression of one student's ability in mathematics. For example, one student (young, male) often tried to "hack" into the working of the program and spent very little time on the set topic. The student attempted about 75% of the preparation for each tutorial but never wrote anything down during the tutorial itself. He indicated that he preferred to work solo. When the computer tutorials began alternating with traditional tutorials, a more detailed assessment indicated that this particular student had extremely poor understanding of the mathematics. He voluntarily dropped out of the course soon afterwards.

Students in the *Computer Group* opted to work alone with their CD-text. After a few weeks there was little if any interaction between students as each worked at their own pace on different aspects. During the one week in which the students were forced to share a CD between pairs, there was increased student discussion, the students took longer to go through the work and there was considerable frustration. Some students wanted to concentrate on the concepts, others wanted to do just the exercises. All the students opted to return to one CD ROM per student.

Students' opinions

The students commented that the main advantages of the CD-text over using the textbook was the instant feedback and ease in finding and understanding concepts that the CD-text gave. The main disadvantage stemmed from the better availability and portability of a textbook. Most of the students found the CD-text easy to use and well set out. When asked why they liked the using the CD-text the main comments were:

- Easy to understand.
- Easy to get started.
- Novelty value.
- Do not need to carry a textbook around.
- Not as dreary as a mathematics textbook.

Incentive to use in addition to tutorials

Throughout the year students had access to the CD-text outside tutorial time. For the first 13 weeks, this access was limited to a maximum of 3 days since the CD was needed for the tutorials. During this time, three students booked the CD on three different occasions. Two of these students were not seen again in tutorials after the 13th week. After the tutorial groups alternated, three students borrowed the CD three times (one of these was a previous borrower), three students borrowed it twice, and five students borrowed it once. The students who borrowed the CD were young and predominantly female. All commented that they found the CD-text easier to understand than either the lecture notes or the textbook.

8.4 Summary

The Textbook task in Section 8A consisted of extended passages of text, interspersed with definitions, theorems and examples. It was presented in a sequential fashion. Students using the textbook tended to learn in a sequential fashion even though they had the option to skip to any section they wished. Therefore, the textbook tended to be read from beginning to end and students backtracked only to confirm or consult an aspect that was relevant to a current passage being read. The textbook authors controlled the order of learning rather than the student. Many found the language in the textbook difficult, especially for comprehension of the theorems. There was also a reluctance to skip forwards to another section that contained examples of applications. Students thought the applications looked more daunting than the section they were working on, indicating a negative reaction to perceived difficult or vast information.

The multimedia approach had the greatest motivational appeal because of the video clips and interactive graphics. The dominant features for understanding mathematical concepts are reported to be diagrams and moving graphics (Tringa & Lipitakis, 1995). However, unlike the younger students, the more mature students were frustrated with the interactive graphics. The difficulty appeared to be the linkage between the underlying mathematical model and the graphics. While the younger students enjoyed changing values that automatically altered graphs, they displayed little or no understanding of the mathematical link to these changes. The more mature students however made attempts to find the linkage and this resulted in their frustration. None of the students using the package could follow the mathematical concepts. Why not? First, although the package contained an option to choose between applications, concepts or exercises, the material was presented in a sequential manner. The video had to be seen first. Students going straight to the concepts option missed initial concepts that were attached at the end of the video clip. Secondly, the concepts were presented in a more abstract form with limited examples, and exercises beyond the comprehension of a student learning new material. Although students were directed to a booklet associated with this package the draw of the computer outweighed the desire to read hard-copy text.

Several other possible influences that relate to recent literature emerged from comments from students working on the Multimedia package. Comments such as "I didn't know where to start" or "It's difficult to know what to go through..." indicates a lack of student self-regulated learning strategies (Young, 1996). In order to work effectively with software that gives the learner choices, the students need to have the skill to realise what is needed and how to find the information. Although Young (1996) found that a programmed-controlled environment was not too dependent on student strategies, a difficulty with the Multimedia task was the sequential nature of the content that gave an impression it was non-sequential. The student self-regulating strategies did not appear to be compatible with the software presentation. This became more pronounced with pairs of students who were not so flexible in adjusting their strategies to fit the software.

Each of these learning resources is analysed in the next chapter in terms of linkages and text layers in an attempt to understand the results of these observations.

Chapter 9 Analysis of Learning Resources

9.1 Introduction

This chapter takes a more detailed look at text in the four different learning resources mentioned in the previous chapter, that is, the textbook, CD-text, text-based interactive courseware 'Epsilon' and a multimedia-based interactive software package. The first two are learning resources in which students do not actively input data. The second two represent interactive learning resources. The analysis of the learning resources outlined in this chapter focuses on the topic used in the comparison study as outlined in Chapter 7, that is,

Second order homogeneous differential equations with constant coefficients.

While acknowledging that the sample size for comparing the four learning resources was too small to generalise beyond the study, there was a strong indication that the CD-text and the interactive software could enhance student comprehension of a topic more than the textbook or multimedia package.

Throughout this chapter some of the metacognitive activities (Haller, Child, & Walberg, 1988) that could account for differences in comprehension between the four learning resources are mentioned. These activities included

AWARENESS. Self-awareness and responsiveness to

- Components, cues.
- Level of comprehension. How much of the topic did they think they understood?
- Text dissonance. Were they aware of conflicting ideas?
- Explicit and implicit ideas.
- Medium (e.g. likes/dislikes about medium).

MONITORING. What activities did they employ?

- Intentionality. Why read the text?
- Self-questioning, summarising, paraphrasing, synthesising. What activities did they use to cope with the text?
- Self-directed activities such as relating details to main ideas, integrating with prior knowledge. Linkages within and outside the text.
- Evaluating activities such as prediction, assumptions, confirming hypothesis.

REGULATING

- Strategies such as re-reading, backward and forward search strategies, contrasting textual information with prior knowledge, comparison of main ideas with each other and with details.

Underlying these metacognitive activities is the basic structure of the text. How did the text structure differ between the four different learning resources? How did structure influence metacognitive activities? For example, what about the ease of linking between sections of text, the layout that encouraged or discouraged reading, or the level of comprehension required between component links?

In this chapter, each of the four learning resources are discussed in terms of the four layers of text as outlined in Chapter 7 (Figure 7.1) and how the students used the linkages. The analyses of the different learning resources address the following:

- Layer 4 involves the linking between major text sections. How did each of the learning resources represent the linkages between the introduction, definitions, theorems, examples and exercises? Did the students sever some linkages and what influences did this have on metacognition and comprehension?

- Layer 3 involves the comprehension of a section of text such as a theorem or definition. For each of the four learning resources, what sections did the students comprehend well and did the learning resource reinforce comprehension?

- Layer 2 incorporates the linkages between basic word, symbol and graphic components. How did the learning resources differ with these linkages? Did the linkages affect metacognition and comprehension?

- Layer 1 involves the comprehension of individual words, symbols or graphics. How simple or difficult were these individual components. Did they influence the metacognition and comprehension of the text?

9.2 Non-interactive learning resources

As mentioned in Chapter 8, non-interactive learning resources are either hard-copy text (textbooks) or software that does not involve direct student input or interaction. Student external actions are confined to the metacognitive activities already mentioned such as note-taking or performing calculations. Software in this section is usually a summarised form of a textbook.

9.2.1 The Textbook

The reading on second-order differential equations was taken from Anton (1995) 'Calculus' (pp. 983-989). The sections were structured by the text author in a specific order:

- Aim (stated in one sentence)
- Subtopic 1: Second order linear differential equations
    * Definition
- Subtopic 2: Linear independence

    * Definition
    * Theorem (and reference to proof in another book by the same author)
- Subtopic 3: Constant coefficients
    * Definition of auxiliary equation
- Subtopic 4: Distinct real roots
    * Definition
    * Example
- Subtopic 5: Equal real roots
    * Definition
    * Example
- Subtopic 6: Complex roots
    * Definition
    * Example
- Subtopic 7: Initial value problems
    * Definition
    * Example
    * Summary
- Exercises 1 - 32.

This approach is presented to the reader in a sequential fashion: the aim, theorem, a reference to where the proof is located, a definition of the three types of solutions accompanied by examples, a summary and finally, a series of exercises for the reader to attempt. The order of the sections is structured by the textbook author, so that substantial effort is needed by the student to find links other than those explicitly displayed. The study in the previous chapter showed that students tried to comprehend the topic by more or less following this order.

For some students aspects of hard-copy text detract from concentrated reading. As the reader turns a page, the eye would skip ahead and the impression gained would be a huge amount of material still to be understood.

> *There is so much to cover, it is really daunting.*

This encouraged weak links and a surface learning approach in text layer 4 between some sections as students skipped or skim read paragraphs with little or no understanding. The bulkiness of the text also appeared to discourage students from seeking relevant information, such as applications, from other chapters outside the immediate text (Chapter 8, Section 8A). On the other hand, students developed stronger links between examples and the summary, resulting in much of the knowledge on the topic being extracted from the examples, summary and exercises. This format did not encourage in-depth concept development through a deeper approach to learning.

The comprehension of sections in text layer 3 varied. Students generally found the theorems and definitions harder to comprehend than the examples and summaries. The theorems generally required high level **symbol-symbol** and **word-symbol** comprehension while the examples and summary required lower level comprehension of linkages between components. For example, consider

THEOREM: If the series $\sum u_k$ *converges, then* $\lim_{k \to +\infty} u_k = 0$.

(Anton, 1995, page 531)

as opposed to:

EXAMPLE:	*The series* $\sum_{k=1}^{\infty} \frac{k}{k+1} = \frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \ldots + \frac{k}{k+1} + \ldots$ *diverges since* $\lim_{k \to +\infty} \frac{k}{k+1} = \lim_{k \to +\infty} \frac{1}{1 + 1/k} = 1 \neq 0$

(Anton, 1995, page 532)

At one end of a spectrum, the procedure in the previous example can be followed with little understanding of the concept. At the other end of the spectrum, the theorem requires thought processes that need to decipher $\lim_{k \to +\infty} u_k = 0$ in terms of summation and convergence.

Within some sections such as the introduction, definition and explanations, an additional deterrent was the high level of comprehension needed in narrative reading. Although the narrative reading should have been easier to comprehend, sometimes it can be the way an idea is expressed that can cause considerable difficulty. For example,

> *A function is called **differentiable on a region R** of the xy-plane if it is differentiable at each point R. A function that is differentiable on the entire xy-plane is called **everywhere differentiable** or simply **differentiable**.* (Anton, 1995, page 801).

The word 'differentiable' has been emphasised in bold by this author to illustrate the number of times it occurred. Once was in relation to a specific region, once in relation to a point and twice to an entire plane. Therefore, the quantity as well as the quality of layer 2 linkages (**word-word** and **word-symbol**) could affect comprehension in layer 3.

The structure within sections at the layer 3 level of text also encouraged linking to previous equations, exercises or theorems. For example:

> *Neither of the functions* $e^{m_1 x}$ *and* $e^{m_2 x}$ *is a constant multiple of the other (Exercise 29), so the general solution of (4) in this case is....*

In this example, the reader is drawn to Exercise 29 at the end of the text as well as to a previous equation (4). It was noted that none of the students followed these links to either the cited exercise or the equation. The written text did not appear to encourage the follow-up of links outside the immediate area of text.

Comprehension for layers 1 and 2 were assisted by text cues. Key words such as *linear dependence, linear combination* and *auxiliary equation* were highlighted in bold within sections and this helped to both draw the attention of the reader and strengthened the awareness of the link between these key words and their symbolic counterparts. Other strategies included critical symbolic equations being highlighted in coloured box outlines and the repetition of key words or symbols. It should be noted that the particular topic being analysed did not contain any diagrams, although limited diagrams are available in other sections of the textbook.

The hard-copy text inspired some monitoring via summarising and the integration of ideas with prior knowledge. Students used pen and paper not only to summarise points but also to repeat the examples and to try a couple of exercises. This monitoring did not apply to theorems as evidenced by the taped conversations between pairs in the study in Chapter 8. When dealing with theorems, the students could not follow the main ideas and predictably could not relate any parts of the theorem to prior knowledge or to the rest of the text. Some notes were taken and there was a lot of re-reading.

Therefore, the textbook reading encouraged some awareness of cues, a tendency for lower level comprehension with examples, as well as some monitoring and regulating. A main feature of reading the textbook was the requirement for the reader to sift through the information to determine the main points and details. Although the hard-copy text helped with cues such as highlighting and colour identifying, comprehension of pertinent information and the determination of dominant or critical linkages was the responsibility of the reader.

9.2.2 CD-text

As mentioned in Chapter 8, the CD-text used here is a standard first year mathematics textbook that has been summarised and placed on a CD ROM. There are some simple graphic animations throughout the program usually found via *exploration* or *project* hypertext buttons. The software analysed in this study was from 'Interactive Calculus', a program based on a textbook by Larson, Hostetler and Edwards (Larson, Hostetler, & Edwards, 1995).

The software program is user-friendly in that the text is short, clear and widely-spaced on each screen with hypertext buttons as access points to further sub-sections. These hypertext access buttons help classify this learning resource as *non-sequential.* For example, on the initial screen for a topic, the students can choose between exploring the general topic, or choosing from optional paths for concepts, examples, exercises or a project. The initial screen for the actual topic being analysed consisted of:

- Short narrative sentence stating the aim of the topic.
- Hypertext button called EXPLORE IT FIRST that gives a 10 line proof for changing complex solutions from cartesian to polar form.

- Three sub-topics with hypertext buttons for concepts, examples and exercises in the following layout:

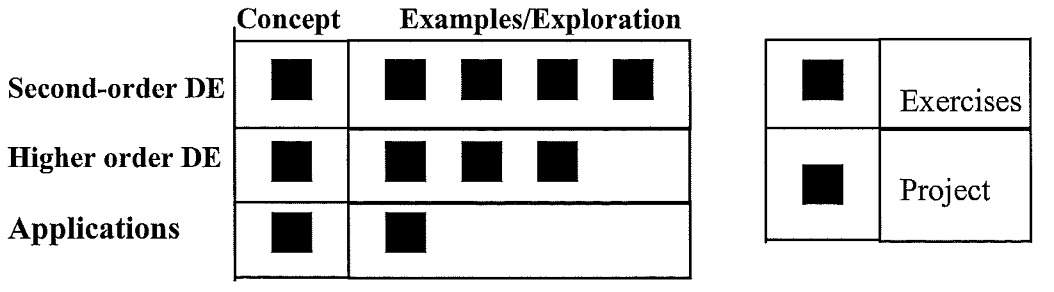


Figure 9.1: Layout of initial software screen for a topic

Each hypertext *concept* button had 3 screens and two of the three screens contained further hypertext buttons for access to definitions and theorems. At the end of the theorem a further hypertext button gave access to the proof of the theorem (Figure 9.2). The hypertext *example* buttons led to one screen containing both a sample question and solution. Each solution was illustrated and linked to a static diagram. If a solution required more detail, parts were packaged together with access via another hypertext button situated in the appropriate section of the solution. Exercises, accessed from the initial screen, contained solutions to odd questions via a hypertext button adjacent to the question.

What does this mean in terms of linkages within layers of text and the influence on metacognitive activities? This non-sequential program structure gave students easy access to many sections and encouraged monitoring activities such as note taking as seen in the comparison study in Section 8A. The structure also meant that the students had ready access to examples and exercises, the sections most preferred by the students using the textbook. An additional feature was that the non-sequential layout also encouraged students to access the concepts. The students' notes in the Section 8A study looked at concepts and not just exercises, thus supporting this argument. The more difficult sections of the topic containing abstract elements, that is, the theorems and proofs, were not readily accessible to the students on the initial screen and could only be accessed by further hypertext links. This was an advantage in that this linkage was accessed when the students felt they were ready to tackle the more difficult aspects of a topic. This behaviour was observed during the study in Chapter 8 where students who initially ignored the hypertext theorem button came back to it after they felt they understood most of the topic. The ease of re-reading, and backward and forward search strategies also encouraged regulating metacognitive activities.

The non-sequential approach effectively divided the topic sections (layer 4) into sub-layers. The main points in concept, examples, exercises and project were in the most accessible sub-layer; the theorems and solutions to exercises were in the next sub-layer and the proof of the theorems were in the third sub-layer as depicted in Figure 9.2.

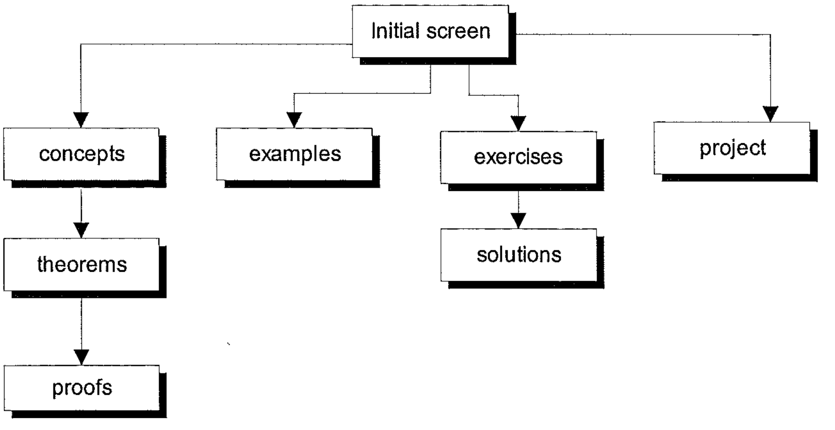


Figure 9.2: Non-sequential sub-layers of text for 'Interactive Calculus'

Each sub-layer was accessed by hypertext buttons. In this way the sections were categorised in terms of difficulty with the more difficult to comprehend being in the lower sub-layers. An added feature was the inclusion of diagrams with each example. This reinforced **symbol-graphic** links and catered for the more visually-orientated students.

Therefore what differed from the textbook was the hypertext connections that reinforced crucial links in helping students to comprehend a topic. These hypertext links were immediately available on request within appropriate sections of text. The hypertext links predominantly guided the linking of text sections in layer 4 and these combined with the summary-type presentation of each section (layer 3) aimed for a higher level of topic comprehension. Examples of strong layer 4 text links included the links between concepts, examples and exercises. Concepts were explicitly linked to both examples and exercises. Exercises were linked directly to detailed solutions. Concepts were linked to definitions and theorems. Theorems were linked to the proofs. All these hypertext links were aimed at strengthening layer 4 linkages. However, these would not in themselves improve comprehension of a topic unless the sections themselves were easily understood. The summary approach of the explanations being limited to the screen size helped to improve comprehension at layers 2 and 3 and this was done by explicitly and visually offering critical linkages to the students. This aspect differed from the textbook where the reader had the responsibility to extract the main points from a larger body of text.

9.3 Interactive software learning resources

Interactive learning resources for this research are software programs specifically designed for self-study purposes. The software allows the student to input answers or perform activities resulting in prompt feedback. This includes direct typing of answers (input fields) and altering values that result in a graph change.

9.3.1 Text-based interactive software

The interactive courseware used in this analysis of a text-based technological learning resource was software developed by Monash University, designed specifically for first year tertiary mathematics (Monash University, 1997). This particular software was released in early 1998 and used in the Chapter 7 study comparing learning resources. The software package covered both calculus and linear algebra topics. Throughout the program are scattered hot words, hypertext buttons, input fields, pop-down fields, pop-up windows and audio clips.

The content in the software was predominantly sequential in that the program author(s) dictated the sequence of reading. However a major difference from the textbook's sequential structure was that many sections were linked to sub-layers via hypertext buttons or hot words. For example, in the topic on homogeneous second order differential equations as outlined in Figure 9.3 the dotted arrows represent the hypertext links. The solid arrows represent the sequence of text screens.

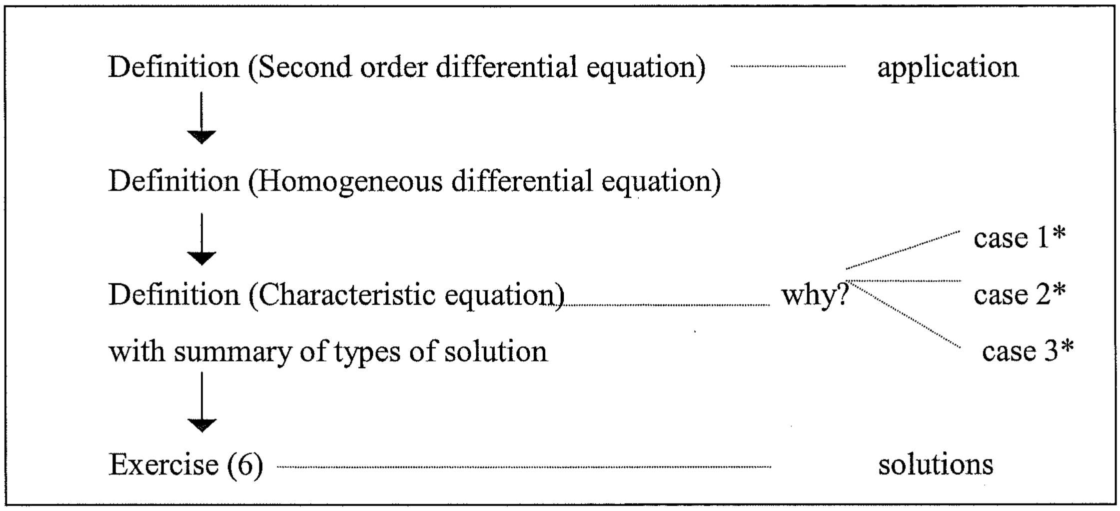


Figure 9.3: Structure of interactive courseware text ('Epsilon')

Theorems were not explicitly stated but were incorporated, with simplified proofs, within the hypertext *why* button. The case examples, shown with * in Figure 9.3 were in numerical rather than general form. There were three exercises on each of two screens and a hypertext button gave the detailed solution. Additional interactive activities displayed in another section used by the students (covering general differential equations) included matching equations to solutions by dragging arrows, matching equations to graphs by choosing from a box of options and obtaining instant feedback in the form of a green tick or a red cross. The software also displayed a hypertext icon that gave continual access to tools such as a calculator, grapher, and formulae for derivatives, integrals and trigonometry.

Like the textbook and CD-text, this software package used the same basic components and incorporated similar strategies to draw attention to critical components and their linkages in layers 1 and 2 of text. New words or important sentences were highlighted, important equations were labelled, and any component links associated with the main ideas were designed to attract attention using both colour and typeface. Therefore for the textbook, the CD-text and Epsilon, the same level of work was being covered and the same strategies were used to draw the reader's attention to dominant links. Likewise both the CD-text and 'Epsilon' were limited by the screen size to a large text point size and up to 5 sentences per screen. The 'Epsilon' content was therefore brief and in summary form with emphasis on the main points in the topic. Easy to follow content presentation gave enough detail to get both an overview of the topic and an understanding of sections. Because the theorem included numerical examples, achievement of comprehension for layer 3 was generally higher than for the textbook but about equal with the CD-text. It should also be noted that the number of component linkages in each screen was relatively low compared to the textbook.

Comprehension of layer 4 text, that is the linking between sections, was enhanced by the learner having ready access to the main points and the choice of access to the slightly more abstract presentations. The sub-layers helped to link definitions with reasons through the *why* hypertext button and to link the solutions to their examples. Although the students complained about the lack of exercises, this did not appear to adversely affect their overall comprehension of the topic.

Therefore this interactive package not only made students aware of layer 1 components and the main points, but it also encouraged monitoring strategies such as summarising rather than copying and an attempt at all the exercises. The students read the screen in a sequential manner indicating that many of the regulating strategies such as re-reading and forward or backward search strategies were dictated by the non-sequential distribution via the hypertext buttons. The package drew the students' attention to critical linkages in layer 2.

9.3.2 Multimedia-based interactive software

The multimedia package was 'Calculus Connections: A multimedia adventure' developed by Quinney, Harding and IntelliPro, Inc. (Quinney, Harding, & Intellipro, 1996). As one purpose of the package was to be a self-study learning resource, it was therefore included in the comparison. This software came with a basic laboratory-type manual that was designed as an additional resource for the package. The software included a video clip, audio clips, write-on fields, scroll bars for text, animated links between equations and graphs and the option to alter a graph by the click and drag of a point.

The structure of the content was presented as being cyclic. The opening screen of each topic had the following:

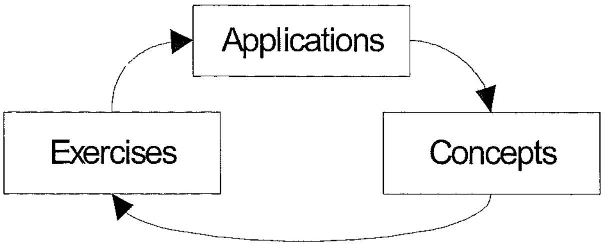


Figure 9.4: Opening topic screen for 'Calculus Connections'

These three sections (Figure 9.4) are sequentially dependent on each other in the order of: applications then concepts then exercises. The link from exercises to applications was not obvious to the students. For example, the applications section modelled a bungee jump using as a second order differential equation, while the exercises included:

> *Show that the phase plane plot of solutions for* $y'' + p^2 y = 0$ *is always an ellipse.*

Not only is the exercise given in abstract form, but the students need to study the appropriate concept section to understand what a phase plane plot was. They could not therefore begin with the exercises, a strategy favoured by many students. The students who did begin with this section switched to the application section after reading the mathematical problem. The exercise was at a similar level to the *project* section in 'Interactive Calculus' and a high level of comprehension was required to understand the full implications.

The only connection the students recognised between exercises and applications was the fact that both referred to second order differential equations. The concepts have clear references to the applications and the exercises were dependent on all the concepts being understood. Some students went first to the exercises and others first to the concepts. All found the reference to previous sections very frustrating. Even within the concept section, sub-topics need to be studied in a sequential manner. The second order differential equation section began with a comparison of the previous sub-topic on first order differential equations and a reference to the application. The connections between the three sections in Figure 9.4 were the program's layer 4 text linkages. The *application* consisted of video clips and animated graphs accompanied by audio clips. Each screen within the *concepts* section had the left half in text and the right half as a graphic. The text itself was predominantly an integrated combination of definition and theorem with numerical examples noticeably absent in the section on homogeneous differential equations selected for the Chapter 7 study.

The main advantage of interactive media is the variety of links explicitly structured into a program. When links are made by the authors, the purpose is to increase the level of comprehension of that component or guide the student so that a crucial link is made. In 'Calculus Connections' crucial links that helped achieve a higher level of comprehension of components (layer 1 of text) were accessed via the hot words that formed a hypertext link between words such as *analytical, derivative, quadratic* and *characteristic equation* and their definitions. The link helped the students to be metacognitively aware of mathematical components, but in this particular program the difficulty was the definition itself rather than the link. Comments from the students indicated that the definitions caused confusion. For example, take the word **derivative**. The hypertext link to this word gave the definition as:

> *The derivative of a function $f(x)$ at a point $x = c$ is the rate of change of $f(x)$ at that point. Writing $f'(x)$ for the derivative of $f(x)$, $f'(c)$ (i.e., the value of $f'(x)$ at $x = c$) is the slope of the graph of $y = f(x)$ at the point $(c, f(c))$. The derivative of a function $f(x)$ is itself a function (also known as the derived function) which gives the rate of change of $f(x)$ as x varies.*

Although the ideas were straightforward, the students commented that by the time they had finished reading the definition they had lost track of the meaning.

The quantity of text per frame was also greater than for both the CD-text and 'Epsilon'. The authors incorporated each frame with text that scrolled to an equivalent of two to three frames. Not only was the text long, but at times the definitions introduced unfamiliar components. For example, in the definition of *homogeneous,* the term **scale equally** was unfamiliar and not explicitly defined.

> *We call the equation homogeneous because all terms in y will scale equally. For example, if we find a solution $y = u(x)$, $y = ku(x)$ will also be a solution.*

Among the physical hypertext linkages were *calculate* or *up-date plot* buttons that directly linked values to a graph at text layer 2 (Figure 7.1). Each frame on a computer screen had a scrolled text on the left side and a graph of the solution to the differential equation on the right hand side. At the end of each scrolled text there was provision for the student to alter variables (within boundaries) for a, b and c in $ay'' + by' + c = 0$. Other screens also gave the students the opportunity to alter initial conditions as well. Once values had been selected a click on the calculate button automatically changed the solution plot to reflect the input values. These linkages represented **word-graphic** and **symbol-graphic** connections. From comments by the students, it was these linkages that required too high a level of mathematical comprehension. The students expressed difficulty and frustration with understanding how the changes in the graph related to the text.

Apart from the physical hypertext linkages that included hot words and buttons, there were more difficult **word-symbol** and **symbol-symbol** linkages the students needed to form themselves. Basically, instead of illustrating concepts with numerical examples, concepts were illustrated in a more abstract form. For example, in describing the third possible type of solution, written in the text was:

> ... *For* $b^2 - 4ac < 0$, *the underdamped case (3), the general solution is*
> $y = e^{\mu x}(A\cos(\upsilon x) + B\sin(\upsilon x))$ *where* $\mu = -\dfrac{b}{2a}$ *and* $\upsilon = \sqrt{4ac - b^2}$.

Or

> ... *we will try* $y = e^{px}$. *We know* $y' = pe^{px}$ *and* $y'' = p^2e^{px}$.
> *Substitution into the equation gives*
> $ap^2e^{px} + bpe^{px} + ce^{px} = 0$ ...

The students found the level of comprehension required for abstract symbols was higher than for numerical examples. It was further complicated with symbols at the layer 1 level of text that were unfamiliar to the students. For example, $\dfrac{d^2y}{dx^2} = f(x, y, y')$. The right hand side of the equation caused the comprehension difficulty.

Therefore the potential advantages of the multimedia program in aiding the student to comprehend mathematics appeared to be lost. A high level of comprehension was required for the physically constructed **symbol-graphic** and **word-graphic** links and the student-constructed **symbol-symbol** links in this particular resource. The program did not encourage monitoring activities such as note taking or summarising. The high level of abstract explanation also reduced the amount of regulating strategies observed in the other three types of learning resources.

9.4 Summary

Many of the text layer 2 linkages were difficult to find in the Textbook and required high comprehension for the multimedia software package. The layer 2 linkages in the CD-text and 'Epsilon' were easier to comprehend and therefore achieved a higher level of comprehension. Since the textbook and Multimedia software obtained low output scores in Chapter 8 while the CD-text and 'Epsilon' scored significantly higher for our sample, then text layer 2 linkages could play a more important part in the development of efficient reading to learn strategies in mathematics. This does not mean that other text layers are not essential, but rather that a focus on easing text layer 2 linkages should help students comprehend undergraduate mathematical text. There is also a need to continually assess which text or self-study software packages are helpful in this respect.

Chapter 10 Conclusion

10.1 Secondary to tertiary mathematics

Research literature highlights some of the major epistomological/cognitive, social/cultural and didactical changes students encounter as they pass from secondary to tertiary study. In mathematics, there are changes unique to the field. These include the transition from practical to abstract thought, the hierarchical nature of mathematical concept development that makes prior knowledge critical for higher learning and the time needed to understand and absorb topics. These concerns are reflected in complaints from students about the unrealistic expectations of prior knowledge by lecturers, the fast pace of lectures, the lack of concrete examples and the concern with theorems and proofs. The overall result is lecturer dissatisfaction, not only with student weaknesses but also with the students' perceived lack of interest in mathematics resulting in inhibited development of intellectual autonomy in mathematics.

10.2 Impact from schools

Some of the current difficulties students experience in first year mainstream mathematics can be explained from the observations and interviews conducted in the senior secondary schools. There have been many impacts on tertiary mathematics including curriculum changes, teaching styles, learning styles, technology, resources and algebraic and problem-solving skills. Two significant current impacts are a lack of skills in reading to comprehend mathematics and weak algebraic skills. Both of these were chosen for further investigation and in essence were negative impacts on undergraduate mathematics.

10.3 Algebraic skills

Algebraic skills were studied at the undergraduate level and five categories of common difficulties emerged from the analysis. Most of these five categories not only included a variety of error types as defined by other researchers such as Davis (1984) but were also more than just "generalised arithmetic" as proposed by Booth (1988). For example, Category 1, (*ability to apply the order of operations agreement, especially the role of brackets*) and Category 2 (*ability to apply the properties of numbers, especially fractions and rational functions*) are the closest categories to 'arithmetic being generalised' and include all possible types of errors that could be made with brackets and rational functions. The other categories, (3, 4 and 5) suggest gaps with higher order skills. For example, Category 3 (*ability to follow the structure of underlying procedures*) and Category 5 (*judgement in exploring the range of possible solutions*) seemed to be associated with strategic or planning mechanisms and judgement skills rather than generalised arithmetic skills. Although category 4 (*false generalisation*) has been treated by researchers as an algebraic skill obstacle, in this study it is linked to inappropriate choice of tools once a problem has been comprehended. Again, at the undergraduate level, this involves skills of strategy rather than procedure.

Algebraic skill difficulties were initially noticed in university test and examination script marking and were initially tolerated under the assumption that the errors, at this level of mathematics, resulted from examination stress or minor slips. The concern now is that some algebraic skills should not be so easily explained away and may indicate far deeper and more fundamental difficulties. The five categories mentioned in this study outline algebra gaps that are now acknowledged as unacceptably consistent and common amongst students who are supposed to have successfully passed their final secondary school mathematics examination. Many undergraduate students not only have significant gaps in their prior algebraic knowledge but these gaps are linked to both early arithmetic and higher level skills.

10.4 The effect of higher learning on algebraic skills

The five categories became the basis of further testing of senior secondary students and Year 2 university students so that a comparison could be made as to whether the algebraic weaknesses automatically improved with higher mathematical learning. It was found that not only did similar types of algebraic difficulties become apparent in consecutive Year 1 university years, but those same difficulties were already present in a similar (although often smaller) proportion of senior secondary students prior to their final year of secondary school. The same difficulties were again seen in Year 2.

Algebraic difficulties became more widespread and consistent going from senior secondary to first year university mathematics, and slightly less widespread (in some categories) going from Year 1 to Year 2 university students. Why is there a decline in algebraic skills at the beginning of Year 1 and an almost similar decline at the beginning of Year 2, after an improvement by the end of Year 1? One possible explanation is that some students could have learned their algebraic skills in an isolated, surface learning fashion with little understanding or extension of concepts. Such students would have difficulty coping with situations in which they have choices in mathematics or are required to link ideas. Combine this with a four month break from mathematics before university begins and students can start the university year with significant widespread gaps in algebraic skills.

Improvement in categories such operations on rational functions, structure of underlying procedures and false generalisation was noted at the beginning of Year 2. This confirmed the improvements in similar categories at the end of Year 1, although some decline existed over the holiday break. However, similar difficulties were still apparent at all levels of mathematics, and overall these algebraic difficulties did not appear automatically to correct themselves at the senior secondary or in first year university levels. The conclusion is that in the long term students do not appear to improve markedly these algebraic skills through the learning of more advanced mathematical concepts. Any improvement appears to be temporary.

Competence in a wide range of algebraic skills is assumed in undergraduate mathematics. It is therefore when students have successfully reached or completed secondary school mathematics that any deficiencies in algebraic skills become apparent. Some consolation is that the gaps students bring to university do not cover all algebraic skills, only certain categories, but it is these categories that do not seem to improve with higher learning.

The evidence in this study confirms Rotman's (1991) proposition that the basics learned in pre-algebra arithmetic and early secondary school could play a crucial role in the mathematical competence of many students at a later date.

A possible remediation - a problem solving framework

The research on algebraic problem solving has mainly involved the study of word problems and the subsequent conversion into algebraic expressions. This study indicates that an algebraic problem solving approach should be considered when teaching algebraic skills and using them at university, particularly since some of the categories found in Chapter 3 involved problem solving skills.

At the late secondary and tertiary level the results of the study endorse Schoenfeld's (1994) proposal of emphasising algebraic skills in problem solving. The approach would involve content-specific problem solving tasks with an algebraic emphasis at the early secondary and primary levels. This would demand teaching different algebraic components as the definition of a problem, generalisation of solutions, planning a course of action, and implementation and evaluation of solutions. Teaching students to approach algebraic concepts in a problem solving way should reduce the main difficulties found in this study and enhance the understanding of basic algebra needed for higher level mathematics.

10.5 Reading to learn in undergraduate mathematics

A second impact from the schools is that few undergraduate students read mathematics to comprehend and learn concepts. This is critical in that much of the undergraduate student learning is outside the formal classroom, predominantly in self-study. A questionnaire, set to determine how students used mathematical reading matter, found that textbooks were predominantly used for assignment and tutorial exercises rather than reading to comprehend concepts. This reflected the secondary school approach to mathematical text. It was not until the students reached their third year at university that they began to read mathematics seriously for study purposes. In this way the development of intellectual autonomy appears to be linked to the development of reading mathematics for study purposes

Chapter 6 looked at *what* the students found difficult and *why* they found it difficult. For example, although between a quarter and two thirds of the students gave positive comments for their set of recommended mathematical text, they usually followed this with reservations. Many students found the mathematical text difficult to comprehend. Some commented that they did not know how to read mathematics.

10.5.1 Text comprehension

For the overall topic, a small section of text, a word, symbol or diagram, there appears to be a range of possible levels of comprehension. Although it is desirable to comprehend at the highest level, most students comprehended undergraduate mathematics at a relatively low level and this became more pronounced with theorems and definitions, the sections considered to be the most difficult for their abstractness and the most crucial for information. In terms of Marton's (1997) work, many students were probably learning as committing meaning to memory rather than learning as understanding meaning (Section 5.4.1).

Expository reading behaviour uses strategies such as forward reading, backtracking, re-reading the same passage and using summary techniques with pen and paper. This study found that the students automatically used most or all of these strategies when they were monitored as they concentrated on understanding the meaning of the text, as seen in Chapter 8. However, if the students were reading an extract without direct monitoring, as in Chapter 6, most resorted to reading mathematics text as if it was narrative prose. Expository strategies were not used and many students employed unsuccessful or poor strategies. Students frequently skipped text as it 'looked too hard' and there was no indication on the extracts themselves that students used strategies such as underlining or summaries to increase understanding.

Many of the students not using any expository reading behaviour believed they had a good understanding of the text. This did not strongly correlate with the author's assessment of their knowledge output, thereby emphasising that expository strategy alone did not mean that the students were successful in comprehending the meaning of text. More successful strategies were employed by few students and these consisted of slow, thorough reading of text with a concentration on theorems and definitions. However even the students employing these strategies admitted to difficulty with comprehending abstract material, especially proofs and theorems.

Appropriate strategies can contribute a small part to the reading of higher level mathematics text. Teaching reading to learn strategies is therefore only the first step. A more important step is to determine where to focus the students' attention within the text.

10.5.2 Mathematical text

The results of a triangulation of studies that included questionnaires, interviews and a large group reading extracts led to the proposal of a model based on sub-layers of text within a mathematical topic. The model portrays mathematical text as a series of highly complex, multi-linkages that contribute to a student's overall comprehension of mathematics. There are four layers of text comprising: basic components (symbols, words and diagrams) at layer 1; simple linkages between components at layer 2; a network of linkages giving subtopics at layer 3; and overall text comprehension with complex networking at layer 4. Each layer is encased within the other as depicted in Figure 7.1 (Chapter 7). Within each layer, components, linkages or networks of linkages can be comprehended at any of Dechant's (1991) six hierarchical levels.

The way a topic is presented may be crucial. Within text there are dominant linkages critical to understanding that often require a high level of comprehension. For skim reading of light narrative text, a student can gain an overview of the topic. This implies that the critical linkages are in the outer text layers (layers 3 and 4). In mathematics, there is not only a greater quantity and combination of critical components and linkages than in narrative text but the critical linkages may also at a different text layer. Dominant linkages in mathematical text appear to be text layers 1 and 2. This may explain why skim reading a mathematical extract did not achieve a high level of overall topic comprehension while slow re-reading of each sentence before moving to the next sentence was more successful.

This nested complexity of text layers and dominance at the inner layers can also account for the wide variety of comprehension displayed by students reading the same text. For example, when students have a low comprehension of a critical mathematical word, symbol or diagram, this can adversely affect their comprehension of a topic. Similarly, low comprehension of a basic component that is less critical to the topic can be skipped over with little effect on topic comprehension.

The implication of this model is that reading to learn mathematics at the undergraduate level, and with its critical linkages in the deeper sub-layers, needs to be considered in a different way from narrative text. Mathematical text needs to be approached by helping students to master layer 2 first, thus directing the students' attention to the core of mathematical text where most of the dominant or critical linkages lie. Such an approach would require the development of active mathematical reading strategies for inner text layers. Since these strategies would be dependent on adequate prior knowledge in mathematics and narrative reading, they would need to be implemented in the senior secondary school or first year at university.

Some undergraduate mathematics reading strategies have already been mentioned in the literature (for example internet MAA articles 1998). However the strategies appear to be implemented without any analysis of how successful they are or why some may be more successful than others. Examples of such strategies include small groups reading pages of text to each other with groups analysing each piece of text in terms of content, form and function and then reporting back. Others involve reading assignments in which students e-mail answers to comprehension questions prior to each class, followed by a discussion of answers in class. Alternatively, students could bring three in-depth questions to each class for discussion or a handout with a series of structured questions that get students to explore relevant ideas prior to class. Only some of these strategies focus on the inner text layers. If the proposal in this thesis is reasonable, the strategies that focus solely on the outer text layers should not be as successful as those that focus on the inner text layers. Further research is needed to take a more detailed look at the concentration of text layers and linkages for a variety of strategies and to connect this with their degree of success in helping students comprehend mathematical text.

10.5.3 Comprehension with technology

There is an increasing trend for mathematical text to be presented as computer software. Although mathematical self-study software is still in its early stages of development a sample of the software available at the time of this study contributed to the knowledge of critical linkages in mathematical text.

The study followed the reading and learning behaviour of 47 students as they undertook a reading task on one of four learning resources: Textbook, CD-text (a textbook on CD ROM), 'Epsilon' (a text-based interactive courseware) and Multimedia (a multimedia-based interactive software package). The time spent on expository reading elements such as forward reading, skipping text, re-reading the same passage, backtracking and frequent use of pen and paper, varied according to the resource. For example, students using 'Epsilon' and the CD-text maximised the use of the non-sequential nature of the resource and those on the Multimedia task spent time on the video clips and interactive graphics. The highest post-test scores came from the students using the CD-text and 'Epsilon', while the lowest scores came from pairs of students using the Multimedia package. The Textbook task scores fell in the middle of the ranking in Figure 8.1 and for this task students appeared to read in a sequential manner with the learning order being controlled by the textbook author. Post-test scores obtained in the study were higher for pairs of students than individuals working on the same learning resource. The only exception was the Multimedia task. Statistically, however, there was no significant difference between individuals and pairs of students using the same resource, again except for the Multimedia task where individuals scored better than pairs. It is acknowledged that these scores give a trend for this sample only and are only an indication of a need for further exploration.

Why the difference in post-test scores between learning resources? Why did the Textbook students score lower than students using the two self-study software packages (CD-text and 'Epsilon') but above the Multimedia package scores? Some of the factors influencing these differences may be seen from the students' opinions of each learning resource. According to the students, the CD-text and 'Epsilon' were non-sequential, user-friendly, easy to follow and gave enough detail to get an overview of the topic. The packages display a text that is brief and in summary form where the learner can choose to access the more abstract presentations. The difference between the two packages from the students' point of view was the availability of worked examples and exercises with readily accessible solutions. This feature was greatly appreciated by the students working with the CD-text but the lack of examples and exercises was a cause of dissatisfaction with 'Epsilon' and the Multimedia package. Although the CD-text and 'Epsilon' did not differ significantly with post-test scores for the sample, the actual ranking in Figure 8.1 indicated slightly better scores for the CD-text.

Students may prefer to approach the content by searching for the overview first (global or holists) or the details first (analytic or serialist). Research at the University of Sheffield with 105 undergraduate students who searched databases showed that cognitive and learning styles could influence how students used computer packages (Wood, Ford, Miller, Sobszyk, & Duffin, 1996). In their research, global (holist) learners were more successful searching out information but also less satisfied with their results. If critical linkages for mathematics are in the inner text layers, then mathematical text should be more compatible with the serialist style than the holist. Software that caters for different learning styles by directing both the holists as well as serialists to the critical linkages should show better success for self-study than the textbook. The CD-Text and 'Epsilon' appear to be better than the textbook in this respect. On the other hand, too abstract an approach too early in the learning of concepts could account for the poor results from the Multimedia task and could explain the frustration for the global learners. The main advantage of the multimedia package lay in the way the video clips described physical applications of the topic. Students using this learning resource excelled in the application question in the post-test.

Surveying the mainstream class as a whole showed that the students were rarely using their textbook for self-study. Perhaps a new learning resource is needed to motivate them to increase their reading, understanding and accuracy. This study has shown that some of the software packages can be better for self-study purposes than the textbook, at least in the short-term. However, not all packages that introduce mathematical concepts are suitable for self-study. Moving graphics, animations and video clips alone do not appear to increase student understanding. A clear indication from this study is the role comprehension of a text plays, that is, how appropriate the readability is to the level of the student, and the importance of choice in order to suit the student's learning style. 'Trendy' technology may not necessarily improve learning, but if it is well designed from the students' learning perspective and thoroughly tested then it can significantly enhance student understanding in mathematical concepts and perhaps be more successful than reliance on a textbook.

It should be noted that using a relatively successful CD-text computer package instead of a textbook for tutorials did not appear to gain test score advantage. The novelty of using the CD-text wore off for many of the students and all, except one student who wanted to stay with the CD-text alone, asked to combine the CD-text with the textbook. Over the academic year, the simplification of the concepts and the easy access to instant detailed solutions was not considered an advantage as the mathematical topics and tutorial questions became more abstract and difficult for the students. One comment from a student indicated that as the topics became more complicated, the time on the CD became too short. The students seemed to prefer to go at their own pace, and as the topics became more advanced they needed longer time than was allocated in a one-hour tutorial. All of this points to a more appropriate use of the CD-text as a self-study resource for some students rather than for use in tutorials.

10.5.4 Comprehension linkages in technology vs textbook

For the small sample of students in the study, the CD-text and interactive software package ('Epsilon') appeared to enhance student comprehension of a topic more than the textbook or multimedia package. If we assume that text layer 2 contains most of the critical linkages for initially understanding mathematical text (see Chapter 9) then this may explain why the students experienced more difficulty with the textbook and multimedia package. For the textbook and multimedia learning resources the layer 2 critical linkages were either not obvious and the students had to actively hunt for them or else they required a high level of comprehension. The former situation applied to the textbook which was often skim read. This implied that students were trying to comprehend either initially or solely in layer 4, the dominant layer for narrative text rather than mathematical text. The layer 2 critical linkages for mathematical text were being missed. The second situation applied to the multimedia package where students struggled to link the diagram changes with the adjacent text.

In contrast, the other two packages, the CD-text and 'Epsilon' actively concentrated on layer 2 linkages. Their non-sequential and summary presentation reduced the level of required comprehension and introduced easy access and choice to the student. The packages also helped the students with layer 3 linkages, even though the overall topic comprehension was left to the students to actively construct themselves. The success of these two packages was evaluated in terms of output and student perception.

10.5.5 Mathematics reading at primary and secondary schools

Reading strategies in mathematics need to draw the students' attention to the essence of mathematics. The strategies themselves are only a first step. In the long-term it is more important to train students to comprehend the main ideas and details of an extract by themselves.

At the primary school level, students learn to read narrative text but there is also a good opportunity to initiate students into learning to read mathematics. For example, Munro (1989) maintains that the following strategies improved underachieving mathematics students within all grades in early mathematics. Students can be taught to:

1. Verbalise and paraphrase mathematical statements.
2. Relate statements to background knowledge.
3. Find the purpose of the statement and classify it as a computation, rule, procedure, formula, etc.
4. Find out if the statement needs to be broken down into smaller segments.
5. Learn how and when to break a statement into segments.
6. Learn the procedures for extracting the main ideas.

Munro (1989) argues that at this level, learning to read mathematics is similar to narrative reading. Based on contemporary models of reading he assumes that learning will occur if:

- The learner perceives the purpose.
- The learner expects s/he can learn the idea.
- The learner acts on the environment by sharing ideas, predicting and experimenting.

- The learner constructs hypotheses and modifies them with subsequent feedback.
- The learner communicates with others.
- The learner observes demonstrations and uses new ideas.
- The learner actively reflects what they did.
- The learner selects from prior knowledge.
- The learner modifies existing knowledge in light of new knowledge.
- The learner masters self-instruction, self-questioning and self-management.
- The learner communicates new ideas through examples, pictures, etc.

Learning to read early mathematics based on Munro's supposition has merit if such an approach is taken. It should focus on initiating students into text layer 2 and aim to improve comprehension levels for critical linkages in this layer. The first difficulty we have in New Zealand is that, although there are well-managed reading programs in the schools, the primary schools are not expected to help students to read to learn mathematics, as evidenced by its absence in the Education Review Office report (ERO, 1997).

However, the Educational Review Office report (ERO, 1997) criticises the secondary schools for not actively teaching students in "reading to learn" in topics other than English. In mathematics, the secondary school textbooks and teaching styles tend to encourage repetition of similar exercises. If anything, this tends to discourage students from comprehending mathematical concepts through reading. Although some of the ideas proposed by Munro (1989) have merit at the secondary school level, they are too simplistic for the mathematical knowledge developed in the late secondary level where concepts are presented in a distinct structure and the language of mathematics becomes more complicated and difficult.

This research supports the need for learning to read mathematics at both the primary and secondary schools, especially the mid to senior secondary levels. It is important that students learn strategies to teach themselves mathematics from both hard copy and technological self-study learning resources. This process should ideally begin with basic strategies in the primary school, perhaps along the lines outlined by Munro (1989). The process should progress to the next level in the secondary schools with a focus on text layer 2 initially as students learn to analyse and synthesise written content.

A good foundation in reading to comprehend mathematics should equip students for what the university expects of them.

10.6 Summary

This study has identified that:

- There are two main current impacts on undergraduate mathematics - algebraic skills and reading to comprehend mathematics.
- Despite the many complaints from lecturers, it is only certain categories of algebraic skills that are at the root of the algebraic difficulties experienced not only in first year mathematics, but also in subsequent undergraduate years.

- Students entering university have difficulty reading mathematics and yet at university they are expected to construct their knowledge and supplement lectures with independent reading of topics.
- New Zealand schools need to teach students to read mathematics from the primary school level with the next stage at the secondary level.
- At the late secondary and early tertiary levels, computer software may be able to assist student in comprehending mathematics if it concentrates on guiding students to critical layer 2 text linkages.

10.7 Future Research

Every aspect of this study leads to more questions than answers. For example,

- The end result of the algebraic skill analysis introduces a new way of teaching algebraic skills in preparation for university. Further investigation is needed to find ways of incorporating an algebraic problem solving approach into undergraduate mathematics and to evaluate its effectiveness.
- More work is needed on the comprehension of mathematical text, an area that is not as yet well researched.
    - ◊ A need to validate the correlations between levels of competence and levels of comprehension.
- Further investigative work is needed on the actual linkages in mathematical text:
    - ◊ The connection between levels of competence and levels of comprehension.
    - ◊ The development of linkages by individuals at various ages.
    - ◊ The influence of social pedagogical, metacognitive and environmental factors and how they affect the quality and quantity of linkage comprehension.
- There is a need to develop and evaluate appropriate reading strategies for comprehension of mathematical text at the senior secondary and tertiary levels that concentrates on text layer 2.
- There is a need to evaluate the effectiveness of mathematical comprehension with the latest technologies, especially in terms of critical linkages and their contribution to mathematical understanding. This study only briefly looked at a few examples that had been developed in the late 1990's. The accessibility of computers and the increasingly affordable cost of software means that software for self-study is likely to replace hard copy text in the near future. How does new technology effect mathematical reading comprehension?

Chapter 11 References

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APPENDIX A

ALGEBRAIC TESTS

MATH 105 TEST March 16, 1996

NAME :..

Answers only in the boxes provided

Each box is worth 1 mark

A formula sheet is attached.

1. Rearrange as indicated in the answer boxes

(a) $3y + 5x - 2 = 0$ | $y =$

(b) $\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}$ | $R_1 =$

2. Simplify

(a) $\left(\frac{2x^3}{3x}\right)^2 \div \left(\frac{x}{3}\right)^2 =$

(b) $2x^2(3x + 4) - 4x(2x^2 + 5) =$

(c) $\frac{\sqrt{x} \times \sqrt[3]{x}}{x^2} =$

(d) Express $\frac{1}{2 + \sqrt{3}}$ in a form that does not have square roots in the denominator

3. **(a)** Factorize

(i) $2x^2 - x - 3$

(ii) $x^4 - 16$

(iii) $x^3y - xy + z^2x^2 - z^2$

TURN OVER

3. **(b)** Expand

(i) $(2x+3)(3x-4)$

(ii) $(x-3)(x^2-2x+5)$

(c) Find the coefficient of x^6 in $(x^2+2)^5$

(d) Complete the perfect square on x^2-9x+8

($\qquad$)$^2 -$

4. **(a)** Solve for x

(i) $(3x-4)(x+1)=0$

$x =$ $\qquad$,

(ii) $(3x-4)(x+1)=-2$

$x =$ $\qquad$,

(iii) $x-4=\dfrac{2}{x}$

$x =$ $\qquad$,

5. Let $u=3+4i$ and $v=1-i$.
Express the following in the form $x+iy$, with x, y real

(a) $u-v=$

(b) $uv=$

(c) $|u|=$

NEXT PAGE

6. **(a)** If $|3-2x| \leq 5$
Find A and B where $A \leq x \leq B$

$\leq x \leq$

(b) $\lim\limits_{x \to 1} \dfrac{x^2+x-2}{x-1} =$

(c) $\sum\limits_{k=0}^{\infty} \dfrac{3}{2^k}$

7. **(a)** If $\sin x = \frac{1}{2}$ and $0 \leq x \leq \frac{\pi}{2}$
then $\cos x =$

(b) Write as a single logarithm
$3ln\,2 - 2ln\,8 + ln\,3$

(c) Simplify $e^{ln\,x} + ln\,e^{x} + ln\,1$

8. Differentiate the following functions.
You do not need to simplify your answer.
(i) $f(x) = 2x^7 - x^{-2}$

$f'(x) =$

(ii) $f(x) = 3x^4 \cos x$

$f'(x) =$

(iii) $f(x) = e^{8x^2+3}$

$f'(x) =$

9. Let $f(x) = x^3 - 6x^2 + 12x - 6$
(i) $f'(x) =$

(ii) $f''(x) =$

(iii) Find the stationary point

(,)

TURN OVER

10. **(a)** $\int (x^3 + 2x^{-1})dx =$

(b) $\int_0^{\pi/2} \cos x dx =$

11. Consider the differential equation $\frac{d^2y}{dx^2} - \frac{dy}{dx} - 6y = 0$.

(a) If $y = e^{kx}$ then

(i) $\frac{dy}{dx} =$

(ii) $\frac{d^2y}{dx^2} =$

(b) Find two values of k such that $y = e^{kx}$ satisfies the differential equation.

$k =$

Maximum: 36 marks

M105 TEST 1

March 13, 1997

NAME:..

Answers only in the box provided.
One mark for each correct answer box.
A formula sheet is provided.
If you have any working, use the space on the right hand side.

---

working

1. 3 - 2 x 5 =

2. $\frac{1}{2}+\frac{2}{3}=$

3. Rearrange $3x+\frac{1}{2}y-6=1$
   into $y=$

4. If $\frac{1}{a}-\frac{1}{b}=\frac{1}{c}$
   then $b=$

5. Express as one fraction
   $\frac{1}{x+1}+\frac{1}{x}=$

6. Simplify as far as possible
   $\frac{ab-a^2b^2}{ab}=$

7. Simplify
   $1-\left(1+\frac{x}{2}-\frac{x^2}{3}\right)=$

working

8. Expand
$(2x+1)\ (2x-1)$

9. Expand
$(2x+1)\ (2x^2-x-1)$

10. Find x when $(2x+1)\ (x-1) = 0$

11. Find x when $(2x-1)\ (x-1) = 5$

12. Find the value of *a* and *b* if
$x^2 - 6x - 15 = (x+a)^2 + b$

$a =$,$b =$

13. Find x when $3 - 8x + x^2 = 0$

14. Factorise $x^2 + xy - xy^2 + xz - y^2z - y^3$

15. Simplify $\dfrac{\sqrt[4]{x}\sqrt{x}}{x} =$

16. If $y = \sqrt{x} + \dfrac{1}{x}$

then $\dfrac{dy}{dx} =$

17. If $y = (2-3x)^3$

then $\dfrac{dy}{dx} =$

working

18. For what value(s) of θ are $\cos\theta = \dfrac{1}{2}$ when $0 \le \theta \le 2\pi$?

19. $\int (x^3 + \sin 3x)\, dx$ =

20 $\int (e^{5x} + \frac{5}{x})\, dx$ =

21. Evaluate $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos x\, dx$

22. Simplify $\ln x + \ln[e^{\ln(e^x)}]$

23. Determine the domain of x, namely $x<a$ and $b<x$, for which $\dfrac{1}{x-2} > -1$

$a =$, $b =$

24. Express $\dfrac{1}{\sqrt{2}+\sqrt{5}}$ in a form that does not have a square root in the denominator.

25 (a). For real x and y, express $\dfrac{1}{x+iy}$ in a form that does not have an i in the denominator.

working

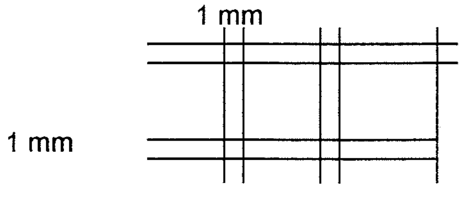
25 (b). Find $\left|\frac{1}{x+iy}\right|$

26. Find a and b such that $y = e^{2x} + e^{3x}$ satisfies the differential equation

$$\frac{d^2y}{dx^2} + a\frac{dy}{dx} + by = 0$$

$a =$ $\quad$ $,b =$

27. Mesh on a pergola needs to be constructed so that 50% of the sunlight is blocked. If the width of the mesh string is 1mm, how far apart should the strings be placed to achieve 50% blockage?
(You may assume the gaps form a perfect square.)



Answer is

END OF TEST

APPENDIX B

QUESTIONNAIRE ON STUDY RESOURCES

QUESTIONNAIRE
WHAT RESOURCES DO YOU USE TO STUDY MATHEMATICS?

You are invited to participate in a research project by completing the questionnaire.
The aim of the project is to determine how you use resources in mathematics.
The questionnaire is anonymous, and you will not be identified as an informant.
You may at any time withdraw your participation, including withdrawal of any information you provided.
By completing the questionnaire, however, it will be understood that you have consented to participate in the project, and that you consent to publication of the results with the understanding that anonymity will be preserved.

BACKGROUND: (please tick appropriate box)

Age:

≤17 ☐ 18-20 ☐ 21-24 ☐ ≥25 ☐

Gender:

male ☐ female ☐

I assess my own mathematics ability as

below average ☐ average ☐ above average ☐ well above average ☐

---

1. How many hours did you spend on mathematics in the last week?
   (EXCLUDE formal lectures and tutorials).

   none ☐ < 2hrs ☐ 2 - 4hrs ☐ 4 - 6hrs ☐ > 6hrs (specify outside box) ☐ ....

2. Excluding formal lectures and tutorials, give a realistic assessment of the amount of your mathematics study time (in the last week) you spent on:

   Revising lecture notes or handouts.................................. ☐

   Using the textbook to understand concepts.......................... ☐

   Using the textbook to do exercises (other than tutorial exercises). ☐

   Doing tutorial exercises or working on assignments.................. ☐

   Other (please specify).. ☐

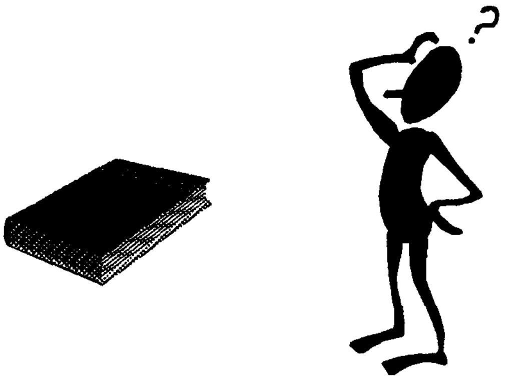
3. What do you think of your textbook?

   ....

   ....

APPENDIX C

READING EXTRACTS



HOW WELL DO YOU UNDERSTAND YOUR TEXTBOOK?

Skim read the questionnaire on the next page to get an idea of what is wanted.

Then read the extract. Aim to understand as much of the extract as possible.

Complete the short questionnaire.

Finally, turn to the back page and answer the questions about the extract. The results will be marked according to the criteria below and will not contribute to your final grade.

0	**did not attempt**
1	**very little understood**
2	**major ideas correct but details sketchy or absent**
3	**major ideas correct and some details given**
4	**major ideas, details and conditions correct.**

Give the completed exercise to your lecturer during this lecture.

QUESTIONNAIRE AND EXTRACT

QUESTIONNAIRE: **Name** ....

READ this first: **Gender........(M or F) Age.........**

Skim read the questionnaire below to get an idea of what is wanted. Then read the extract. Aim to understand as much of the extract as possible. Finally return to this page and complete form below. Feel free to write on the extract.

This extract is about **NEWTON'S METHOD**

Have you encountered this topic before? **YES NO** (circle answer). If yes, where?........................

Use the section indicators ($a \rightarrow h$ and diagrams *D1* and *D2*) located in the margins of the extract to answer the following:

	a	*b*	*c*	*d*	*e*	*f*	*g*	*h*	*D1*	*D2*
The section of the extract you **first** read was ....(tick one)										
The sections you **concentrated on** were...(tick boxes)										
The sections **easiest** to understand were ......(tick boxes)										
The sections **hardest** to understand were ....(tick boxes)										
The most **important** sections for **understanding** the main ideas were..										

How did you read the extract? Did you read whole or part of the extract once, twice, etc?

....

....

....

I understood the main ideas in the extract. (Circle your answer)

Strongly agree — agree — undecided — disagree — strongly disagree

---

EXTRACT BEGINS : **<u>NEWTON'S METHOD</u>**

In beginning algebra one learns that the solution of a first-degree equation $ax + b = 0$ is given by the formula $x = -b/a$, and the solutions of a second-degree equation $ax^2 + bx + c = 0$ are given by the quadratic formula. Formulas also exist for the solutions of all third- and fourth-degree equations, although they are too complicated to be of practical use. In 1826 it was shown by the Norwegian mathematician Niels Henrik Abel* that it is impossible to construct a similar formula for the solutions of a *general* fifth-degree equation or higher. Thus, for a *specific* fifth-degree polynomial equation such as

$$x^5 - 9x^4 + 2x^3 - 5x^2 + 17x - 8 = 0$$

it may be difficult or impossible to find exact values for all of the solutions. Similar difficulties occur for trigonometric equations such as

$$x - \cos x = 0$$

as well as equations of other types. For such equations the solutions are generally approximated in some way, often by the method we shall now discuss.

a

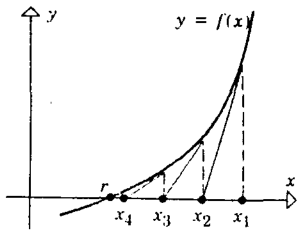


Figure 4.8.1

D1

We note first that the solutions of $f(x) = 0$ are the values of x where the graph of f crosses the x-axis. Suppose that $x = r$ is the solution we are seeking. Even if we cannot find the value of r exactly, it is usually possible to approximate it by graphing f and applying Theorem 2.7.10 to estimate where the graph crosses the x-axis. If we let x_1 denote our initial approximation to r, then we can generally improve on this approximation by moving along the tangent line to $y = f(x)$ at x_1 until we meet the x-axis at a point x_2 (Figure 4.8.1). Usually, x_2 will be closer to r than x_1. To improve the approximation further, we can repeat the process by moving along the tangent line to $y = f(x)$ at x_2 until we meet the x-axis at a point x_3. Continuing in this way we can generate a succession of values $x_1, x_2, x_3, x_4, \ldots$ that will usually get closer and closer to r. This procedure for approximating r is called ***Newton's Method***.

b

D1 (diagram)

To implement Newton's Method analytically, we must derive a formula that will tell us how to calculate each improved approximation from the preceding approximation. For this purpose, we note that the point-slope form of the tangent line to $y = f(x)$ at the initial approximation x_1 is

$$y - f(x_1) = f'(x_1)(x - x_1) \tag{1}$$

If $f'(x_1) \neq 0$, then this line is not parallel to the x-axis and consequently it crosses the x-axis at some point $(x_2, 0)$. Substituting the coordinates of this point in (1) yields

$$-f(x_1) = f'(x_1)(x_2 - x_1)$$

Solving for x_2 we obtain

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} \tag{2}$$

c

The next approximation can be obtained more easily. If we view x_2 as the starting approximation and x_3 the new approximation, we can simply apply (2) with x_2 in place of x_1 and x_3 in place of x_2. This yields

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} \tag{3}$$

provided $f'(x_2) \neq 0$. In general, if x_n is the nth approximation, then it is evident from the pattern in (2) and (3) that the improved approximation x_{n+1} is given by

d

Newton's Method

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \quad n = 1, 2, 3, \ldots \tag{4}$$

e

Example 1 Use Newton's Method to approximate the real solutions of

$$x^3 - x - 1 = 0$$

Solution. Let $f(x) = x^3 - x - 1$, so $f'(x) = 3x^2 - 1$ and (4) becomes

$$x_{n+1} = x_n - \frac{x_n^3 - x_n - 1}{3x_n^2 - 1} \tag{5}$$

f

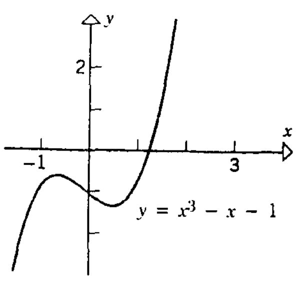


Figure 4.8.2

D2

From the graph of f in Figure 4.8.2, we see that the given equation has only one real solution. This solution lies between 1 and 2 because $f(1) = -1 < 0$ and $f(2) = 5 > 0$. We shall use $x_1 = 1.5$ as our first approximation ($x_1 = 1$ or $x_1 = 2$ would also be reasonable choices).

Letting $n = 1$ in (5) and substituting $x_1 = 1.5$ yields

$$x_2 = 1.5 - \frac{(1.5)^3 - 1.5 - 1}{3(1.5)^2 - 1} = 1.34782609$$

(We used a calculator that displays nine digits.) Next, we let $n = 2$ in (5) and substitute $x_2 = 1.34782609$ to obtain

$$x_3 = 1.34782609 - \frac{(1.34782609)^3 - (1.34782609) - 1}{3(1.34782609)^2 - 1} = 1.32520040$$

g

D2 (diagram)

If we continue this process until two identical approximations are generated in succession, we obtain

$x_1 = 1.5$	$x_4 = 1.32471817$
$x_2 = 1.34782609$	$x_5 = 1.32471796$
$x_3 = 1.32520040$	$x_6 = 1.32471796$

At this stage there is no need to continue further because we have reached the accuracy limit of our calculator, and all subsequent approximations that the calculator generates will be the same. Thus, the solution is approximately $x \approx 1.32471796$. ◄

h

EXTRACT ENDS

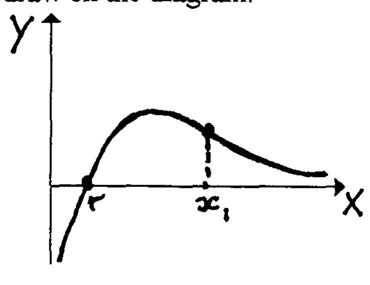
COMPLETE THE QUESTIONNAIRE, THEN TURN OVER

Use Newton's Method to solve $x^3 - 6 = 0$ to 5 decimal place accuracy. You will need to use a calculator.

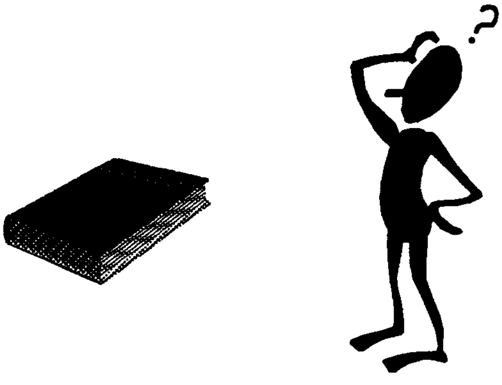
In no more than one or two sentences, explain what Newton's Method is all about (i.e. why use it and how is it used).

When would you use Newton's Method and under what conditions would Newton's Method not work? Elaborate.

In the diagram below, if we wanted to find the root, r, and we began with x_1 as indicated on the diagram, would we eventually get close to the root (r)? Elaborate on your answer. Feel free to draw on the diagram.



In your own words explain what is meant by: $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$, $n = 1, 2, 3 \ldots$



HOW WELL DO YOU UNDERSTAND YOUR TEXTBOOK?

Skim read the questionnaire on the next page to get an idea of what is wanted.

Then read the extract. Aim to understand as much of the extract as possible.

Complete the short questionnaire.

Finally, turn to the back page and answer the questions about the extract. The results will be marked according to the criteria below and will not contribute to your final grade.

0	**did not attempt**
1	**very little understood**
2	**major ideas correct but details sketchy or absent**
3	**major ideas correct and some details given**
4	**major ideas, details and conditions correct.**

Give the completed exercise to your lecturer during this lecture.

QUESTIONNAIRE AND EXTRACT

QUESTIONNAIRE: **Name** ....

READ this first: **Gender........(M or F) Age.........**

Skim read the questionnaire below to get an idea of what is wanted. Then read the extract. Aim to understand as much of the extract as possible. Finally return to this page and complete form below. Feel free to write on the extract.

This extract is about **L'HOPITAL'S RULE.**

Have you encountered this topic before? **YES NO** (circle answer). If yes, where?........................

Use the section indicators ($a \rightarrow g$) located in the margins of the extract to answer the following:

	a	*b*	*c*	*d*	*e*	*f*	*g*
The section of the extract you **first** read was ....(tick one)							
The sections you **concentrated on** were...(tick boxes)							
The sections **easiest** to understand were ......(tick boxes)							
The sections **hardest** to understand were ....(tick boxes)							
The most **important** sections for **understanding** the main ideas were..							

How did you read the extract? Did you read whole or part of the extract once, twice, etc?

....

....

....

I understood the main ideas in the extract. (Circle your answer)

Strongly agree agree undecided disagree strongly disagree

EXTRACT BEGINS : <u>**L'HOPITAL'S RULE**</u>

In each of the limits

$$\lim_{x \to 2} \frac{x^2 - 4}{x - 2} \quad \text{and} \quad \lim_{x \to 0} \frac{\sin x}{x} \tag{1}$$

the numerator and denominator both approach zero. It is customary to describe such limits as ***indeterminate forms of type* 0/0**. As we shall see, a limit of this type can have any real number whatsoever as its value or can diverge. The value of such a limit, if it converges, is not generally evident by inspection, so the term ''indeterminate'' is used to convey the idea that the limit cannot be determined without some additional work. Because geometric arguments and the technique of canceling factors apply only to a limited range of problems, it is desirable to have a general method for handling indeterminate forms. This is provided by ***L'Hôpital's* rule***, which we now discuss. *a*

10.2.1 THEOREM (***L'Hôpital's Rule for Form 0/0***). *Let* lim *stand for one of the limits* $\lim\limits_{x \to a}$, $\lim\limits_{x \to a^+}$, $\lim\limits_{x \to a^-}$, $\lim\limits_{x \to +\infty}$, *or* $\lim\limits_{x \to -\infty}$, *and suppose that* $\lim f(x) = 0$ *and* $\lim g(x) = 0$. *If* $\lim [f'(x)/g'(x)]$ *has a finite value L, or if this limit is* $+\infty$ *or* $-\infty$, *then*

$$\lim \frac{f(x)}{g(x)} = \lim \frac{f'(x)}{g'(x)}$$

b

REMARK. There are some hypotheses implicit in this theorem. For example, in the case where $x \to a$, the statement

$$\lim_{x \to a} \frac{f'(x)}{g'(x)} = L$$

requires that f'/g' be defined in some open interval I containing a (except possibly at a). This implies that f and g are differentiable and $g'(x) \neq 0$ in I (except possibly at a). Similar hypotheses are implicit in the other cases.

c

In essence, L'Hôpital's rule enables us to replace one limit problem with another that may be simpler. In each of the following examples we shall employ the following three-step process:

Step 1. Check that $\lim f(x)/g(x)$ is an indeterminate form. If it is not, then L'Hôpital's rule cannot be used.

Step 2. Differentiate f and g separately.

Step 3. Find $\lim f'(x)/g'(x)$. If this limit is finite, $+\infty$, or $-\infty$, then it is equal to $\lim f(x)/g(x)$.

d

Example 1. Evaluate $\lim\limits_{x \to \pi/2} \dfrac{1 - \sin x}{\cos x}$.

Solution. Since

$$\lim_{x \to \pi/2} (1 - \sin x) = \lim_{x \to \pi/2} \cos x = 0$$

the given limit is an indeterminate form of type 0/0. Thus, by L'Hôpital's rule

$$\lim_{x \to \pi/2} \frac{1 - \sin x}{\cos x} = \lim_{x \to \pi/2} \frac{\dfrac{d}{dx}[1 - \sin x]}{\dfrac{d}{dx}[\cos x]} = \lim_{x \to \pi/2} \frac{-\cos x}{-\sin x} = \frac{0}{-1} = 0 \quad \blacktriangleleft$$

e

Example 2. Evaluate $\lim\limits_{x \to 0} \dfrac{e^x}{x^2}$.

Solution. We have

$$\lim_{x \to 0} e^x = 1 \quad \text{and} \quad \lim_{x \to 0} x^2 = 0$$

so the given problem is not an indeterminate form of type 0/0 and consequently we cannot apply L'Hôpital's rule. By inspection

$$\lim_{x \to 0} \frac{e^x}{x^2} = +\infty \quad \blacktriangleleft$$

f

WARNING. Applying L'Hôpital's rule to limits that are not indeterminate forms can lead to erroneous results. As an illustration, two applications of L'Hôpital's rule in the preceding example would have led to the *incorrect* conclusion that the limit is 1/2.

g

EXTRACT ENDS

Complete the questionnaire, then turn over

Use L'Hopital's Rule to evaluate $\lim\limits_{x \to 3} \dfrac{x^2 - 9}{x^2 - x - 6}$

In no more than one or two sentences, explain what L'Hopital's Rule is all about (i.e. why use it and how is it used) .

Under what conditions would L'Hopital's Rule not work. Elaborate.

Is the following statement correct / incorrect (circle your answer). Give your reason and any corrections (if it is incorrect).

$$\lim_{x \to 0} \frac{\sin x}{2x + 3} = \lim_{x \to 0} \frac{\cos x}{2} = \frac{1}{2}.$$

In your own words explain what is meant by: $\lim \dfrac{f(x)}{g(x)} = \lim \dfrac{f'(x)}{g'(x)}$

APPENDIX D

TRANSCRIPTS OF INTERVIEWS
For Section 6.3.3

INTERVIEW 1 Student A: Female

He didn't explain it [complex numbers] well in lectures. I had to look it up in a book. Its still not ...

Tell me what you now know.

Its basically a normal xy and instead of a y you assume it is a complex number. And then you draw it basically, your complex number has a normal part and a complex part. The normal part is part of the x axis and the complex part is the y axis.

You got that from where?

The textbook. Not from the lectures.

As you progressed through the topic, any parts you found difficult?

Yes, I found polar form difficult. I do not know what it is. I had to look again into the textbook and I must admit I can do it...I can put it into polar form but I don't quite understand what the word means. I assume it means that it is a way of drawing it, but...yea, I only know the recipe basically for that one. Not what it means.

Anything else?

Um...I really got stuck on roots as well.

Can you do it now?

Yes, because I found this formula in the book. So, I know what to do now, but again I don't know exactly what it means. I follow the recipe but I don't exactly know what it means.

Are you quite happy just knowing how to do it?

Um...Usually, if it was like integration I try to make sure I know what I am doing. But complex numbers is just something we did for a week. I am OK with just knowing the recipe and not understanding what I am doing. Because we just touched on it [in lectures].

Can you see the whole picture of where complex numbers fit into the number system?

....I think, ..don't you use it when you take the square root of a negative number.... I find it strange. Somehow you can't take the square root of a negative number, then they develop this theory about complex numbers....

What is the relationship between real numbers and complex numbers?

Nope. I don't know the connection.

What about series?

Um... In lectures we started off doing sequences. I haven't actually revised series yet, so I don't know much. The first two lectures were fine and then I lost it. So again I read the book and I thought it made a lot more sense in the book. ....It was they way you write it down. You know, sigma and the strange little formula that comes afterwards. I did not know what it meant and how to develop the series from there.

What about $\sum \frac{x^k}{k!}$?

Well, that is a general trend for that series and depending on where it starts you just put whatever k is into k. The first term would be $\frac{x^0}{0!}$, the second term would be $\frac{x^1}{1!}$, and so on.

How do you see this. Is it a recipe where you just put in the values or it is something else?

I see it as a general form of a series. In the end I really see it as a series of maybe n terms. Because mathematicians don't want to write them all out they put it into a general form. You can develop from there as many terms as you want. It is a whole line of numbers.

What does sigma mean?

The sum, isn't it? The sum of all the terms. I visualise it as a string.

Anything else?

With the series, I found it quite hard to pick the right tests. Like he have us some tests, I could do the ratio test for any with factorials in it, but for all the other ones I found it really hard to decide which ones to use so sometimes, I would try one to get started, then I would try another one.

Why are you finding it difficult?

I suppose because I am use to doing recipes. There isn't a recipe for that. So maybe you need lots of experience to know which ones to use. .... I have done a limited number of examples, so I haven't got the experience to see it straight away. It is like integration, everything looked cryptic, but after a while you can tell which ones to use. If you do a lot of problems and you start seeing it. You don't really get this from just reading.

Looking further back, do you see the whole topic and where it fits in?

Well no. If we take the series, people want to know if it converges or diverges. It is a very important point. I found it a bit strange because we spend lots of time doing this. But I don't know. It must be very important for something. And there are so many tests for finding it out. Why is it? Why do they want to know?

INTERVIEW 2: Student B: Male

For series, we are supposed to have them memorised, am I right?

What do you say that?

Because when we are doing the review right now he is going over some things and I get the idea that we are supposed to memorise everything. Like the things that can either diverge or converge. I can basically see by looking at it what it is going to do. Because it is a limit and I can basically see what it is going to do. I get stumbled sometimes actually proving whether it is going to converge or diverge.

Why?

Probably not enough practice. I haven't done the homework for this yet. I can basically recognise geometric. Any one of them minus the one before it and do that twice in the whole series.

So what are you looking at?

A pattern. [Students illustrates by writing an examples]. I know that it has a radius of convergence of 1. Um.. and if it.. we need to look at what happens at one and minus one. I figure if it is alternating it diverges.

When you first started your series. Could you read what the lecturer wrote on the board?

It is just remembering how to put the right things in. [students open lecture notes to the first lecture].

At the end of the first lecture, what did you pick up and what did you not understand?

One thing about this whole thing is that he [lecturer] is not very visual in his presentation. Like graphics. To him, he works in x's and numbers. The book also does not have many graphics. A picture helps clear things up for me. I am used to it. If we try to approximate a curve, that is useful. But if we are trying to find x and a bunch of theoretical stuff, it is useless.

What do you call theoretical? Can you give me an example.

If you go through any page, often there does not seem to be any diagrams in it, for ages. Where is the diagram?

I see [from the lecture notes] that he gives theorems, definitions and a general form.

Right. I could follow it but it is just hard slog. Sometimes a good diagram can just crystalise it. You know what I mean? The whole point.... and you don't even have to remember because you know what is happening [from the diagram], and you can make it up yourself. And you don' t have to memorise it. It is more a concept rather than rote learning. Unless he goes too fast.

Did you pick up the meaning of some of the mathematical words such as convergence?

Oh yea.

What about the meaning of the whole topic?

Well, there are only two that you can find the sum one is geometric and one is collapsing. Otherwise you can't find the sum. You just know whether it converges or diverges.

What about $S_n = \frac{a}{1-r}$? Did you know where that came from?

It comes from here [student points to section of lecture notes].

Did you understand it in detail?

Yea, I did at the time. I would have to look through it again. Like, connect it all up again. I don't really know why. I could follow it.

Explain to me the bit you could follow.

Well..... um...... No, I would have to look through it again.

What about the rest?

Divergence test tends to be one of my first tests because there is no point in trying to find convergence if it diverges. The integral test can be one of the harder ones for me because it does not jump out at me that this can be done by the integral. I don't recognise that when things get big that they are still simple. I have to took at it for a while to recognise that all of this is pretty easy. The comparison test is easier, because diagrammatically it make sense. You are trying to get one that is bigger than one you know if you think it diverges.

Do you build up a diagram image of the concept?

No, I don't think so. I can never see that there is anything in my mind. It just makes sense to me if I can just jot down a diagram. I can't see it, but if I write it down I can see it. I also read a diagram. Its a concept rather than a picture. The ratio test is quite easy too. The only thing I have had to remember is that it is the limit.

Why?

Why it is the limit? Because you are looking at it as it goes way up. You are not looking up close, you are looking as far out as you can.

What about the integral test?

Well the integral test is just to remember to replace the k with an x and the sigma to an integral sign. And we need to replace the infinity with a letter and a limit on the outside. You are doing that because you cannot integrate at infinity.

Do you have any difficulties if the work is in general form?

I can if its fast, think and heavy. I get lost. If it comes along with something I am familiar with and it is built up, it is OK. Textbooks can be like this.

What about new symbols or words?

Sometimes if the lecturer uses a new symbol I would just write out the words for it. I found that then I would forget what that symbol means real easily and I am only listening while I write it everytime the lecturer uses it. Some teachers get so full of the little symbols for everything that you just can't keep up.

If you were reading to revise, would you read every single word or symbol?

Um... I would have to read every word if I was trying to understand it. Unless I knew it and I was just trying to remember the key points. But if it has been a while since I have seen it, well, I need to concentrate on every bit.

When you are studying do you stop periodically or skim over the work?

I'll stop at every sentence at least. As soon as I hit something I don't understand I have to keep going over it until it begins to make sense. I would keep going back to the beginning of the sentence.

What about reading in the textbook?
Well, as soon as I open a book, the first thing that hits me is the diagrams. [fig 11.9.1] This makes sense to me. There aren't very many of these. You can see what it is doing and you can see what it is looking for.

Did you read the narrative part yet?

No, I just get everything from the diagram. I was disappointed the whole way through the text because there is not enough diagrams.

INTERVIEW 3: Student C: Male

I am not a good textbook reader. I learn by getting into problems and doing examples. It[doing examples] forces me to go to look at relevant notes, relevant parts of the text. An the more I do that, the easier it becomes. I knew I wasn't a good exam person.

Take sequences and series, you have come out of lectures, are there any aspects that you didn't understand?

I usually find that I can follow lectures. I don't like not following a lecture. I will ask questions in class.

What about the more abstract?

There was a stage where I would just write it down, I would look at the symbols and say, what is going on here? And...series is a bad example, because I didn't do the tut prep on series because it coincided with my COSC assignment. And... I skipped about 3 lectures as well just trying to get that assignment done.

Have you caught up on the work yet?

Um...the bits that we went through the other day. The examples we have subsequently done in class... I think I can basically handle a Taylors series now. It wasn't until we did the radius one today that ....

Are you following examples or looking at why things are happening?

Well... with an example, I will just follow it. As for seeing an overall reason I would have to look for one. I am going to have to get in and do the tutorial questions. My approach is to effectively to start with what I missed and selectively work through. I go back when I need to. I have a textbook there. As for sitting down and reading a textbook....no.

If the textbook is sitting there, what section do you refer to?

Usually I look for a worked example.. if I can't find an exact ...a similar example, I see which I can apply to this situation even though it is a different one. And...there isn't too much we can't simply plug in.

What about if it was a difficult question with no similar example?

I would look at the bits... a good example was that limit question we got from the lecturer. There was nothing relevant to that directly in the series. It was just a case of lets go to every little bit in this expression and see how we can do it in other ways.

Did you do this with the topics generally?

Um... Probably, ..when we get to the end of a topic, the sorts of example given then tend to be one that puts bits together. The last week in every topic tends to have the bits that draws things together. But I wouldn't consciously go looking for that. This is what we are doing... ah! I can see that. I do not set out deliberately to try and understand a topic. But in the end I took all of those little bits and sections and put them together in my mind. It makes sense then.

What about some thing you have done before a long time ago?

Um.. usually if I haven't done it recently, I have to go back and remind myself. Usually you have to kick start yourself. If it was the determinant one it would be just.. I would have to check the textbook for exactly how to do it. I would have to nut out the problem again.

Does it take you a long time?

If we take the assignment question that I mentioned before, it took me longer because I basically ignored the third part of the question.

Why did you ignore it?

Because I can do that. And of course if I had gone through in progression and actually done that part it would have given me the next step far more easily. But just looked and said, "I know I can do that", so I went onto the next bit. The early bit I did because I could see at a glance that I could do it. But there was a bit within it that I didn't see. If I had seen its relevance ....

What were you looking at?

It had ......the expression was similar, the symbols were similar. For example, today in lectures it was $\sum \frac{1}{k^2 - 1}$. Show that this converges. And I though he did it by saying that you don't touch the ratio test when it is a polynomial over a polynomial. So I looked at that and I thought to myself, well, that is similar to ....a/1-r, then I thought lets make a = 1 and r = k2. And I looked at it in terms of a geometric series, which is not what he used. So I connected with something that I knew. I can see why he did the comparison, but I haven't had time to try the geometric. We've had things in physics where we try to get n isolated and there is no way. So I tried to use series. There were many times where I would use a little bit of knowledge and getting myself deeper and finding that it was worse. None worked easily. They suggested trial and error.

Anything else about any other topics?

When we get into absolutes and limits and continuity I get stuck. But I will just have to get in and do lots of examples.

INTERVIEW 4: Student D: Male

I have some 7th form textbooks that I use at home. Pretty much all year I have used all my time just to catchup with what we are doing. I have been reading over the lecture notes after.

Any parts you found difficult?

I found that the general forms are difficult. When I come to do homework problems I look for worked examples to help me along the way. I have done that from lecture notes and the textbook. I have found that the textbook at times can be terrifying. There seems to be big jumps in the working. I found the second order de easier than the first order. The tutorials are a big help.

How do you use the textbook if you had it open in front of you?

Um... when I was study for exams , I basically went through the notes. And then finding examples to do from the textbook. There are certain things that I work on, and then I think, they must be in this chapter because I have to work on it. But sometime I can't find it. At times I just give up. At other times, I hit a wall. I leave it and come back to it to the next day.

Is it the more abstract or examples.

I concentrate on the examples. I read the rest, especially the definition. Some of it I can understand, but the more general proofs I can't understand. I find I try to read it thoroughly because I want to find out the nuts and bolts of it. But this here, doesn't make a great deal of sense. It is not user-friendly. I get a bit messed up with it.

Do you read the definitions, theorems?

No, some of these in that general form, I find strange as to how they get there. I find it quite difficult.

What about the examples?

I find them quite easy to follow. If there is a worked example, that tends to be where it helps me work through. Especially the examples near the end of a chapter. And sometimes I don't understand how to do them. I also do try to go back through the notes but sometimes there are lots of jumps in the notes. For example, the de stuff, I rely on the worked examples.

What about something that you don't recognise?

I try and ask other people for help. But generally the worked examples are not too bad.

Do you every try to work out what is going on?

That's what I try to do when I read through the text, but I usually can't see the path for the trees. I am surprised the text does not explain things more explicitly so that you can understand it.

What about the jargon the text or lecturer uses?

That stuff is not too bad. For example, exponential decay and doubling time. I don't have a problem with that.

What about diagrams?

I skip over these, or perhaps I may glance at them...I,...yes, I skip them.

What about series?

I can follow what is happening, but... I can't appreciate why we want to know these things. The lectures are not clear.

[Student takes lecture notes out]

Tell me what parts you find difficult to understand

Well basically the first bit is ok, but I don't really follow the bounded part. I can see where they are getting at. The limit between these two values. But I can't see how the diagram fits with the symbols.

What about the theorem?

I can go back and read it, but I can't tell you what it [the theorem] was about. I get quite bogged down. I don't know whether I need to sit down and look at it a lot harder. I don't usually. I just skip over it. I can't know what this means. I think this means it can't go onto infinity. If that has reached that limit… I don't know exactly what it means Probably I am a bit guilty of switching off a bit if I can't follow some of that more general stuff.

Do you just skip over it?

I do try and read to understand it. But I still can't get around it. I can sometimes understand what they talk about, but the absolute nitty gritty , no.

Do you feel at the end of a topic, you understand what is going on.

If there are lots of examples and exercises. In this [points to lecture notes] I didn't have anything really to draw on. I understand some areas such as L'Hopital but I have trouble with a lot of others. I can sort of see what they are on about, but I have a bit of trouble with the general formula. I have a lot of trouble with proofs.

APPENDIX E

DATA FOR READING EXTRACTS (CHAPTER 6)

ANALYSIS OF BOTH NEWTONS AND L'HOPITALS EXTRACT

Note: Extra analysis of data is given here for the reader's interest

L'Hopital's Rule (n=189) Newton's method (n=185) Total (n=374)

How often do students use text to understand concepts?

	L'Hopital's	Frequency	%
1-NEVER	1	1	0.53%
2=RARELY	2	14	7.41%
3=SOMETIMES	3	55	29.10%
4=OFTEN	4	66	34.92%
5=ALWAYS	5	19	10.05%
	no answer	34	17.99%
		189	100.00%

Newton's	Frequency	%
1	1	0.54%
2	8	4.32%
3	56	30.27%
4	63	34.05%
5	19	10.27%
no answer	38	20.54%
	185	100.00%

	average %	TOTAL
1	0.53%	2
2	5.88%	22
3	29.68%	111
4	34.49%	129
5	10.16%	38
no answer	19.25%	72
	100.00%	374

REASONS given for why students do/not use their textbook

1 Textbook is easier to understand than lecturer/lecture notes
2 Textbook gives more details than lecture notes
3 Textbook supplements lecture notes
4 Textbook give better or more examples
5 Textbook is only used if 'stuck' on something
6 Textbook is only used for tutorial or assignments
7 Preference for obtaining help from others rather than the textbook
8 Textbook is difficult to understand
9 Cannot be bothered with mathematics or their textbook

L'Hopital's: Frequency

REASONS	Never	rarely	sometimes	often	always	no reply
1	1	4	4	3	2	
2		5	15	17	3	
3		3		14	9	
4		1	3	10	3	
5		1	5	6	1	
6			6	6	1	
7			16	6		
8			3			
9						
no answer			3	4		
	1	14	55	66	19	34
					Total	189

L'Hopital's: Percentage

REASONS	Never	rarely	sometimes	often	always
1		28.57%	7.27%	4.55%	10.53%
2		35.71%	27.27%	25.76%	15.79%
3		21.43%	0.00%	21.21%	47.37%
4		7.14%	5.45%	15.15%	15.79%
5		7.14%	9.09%	9.09%	5.26%
6		0.00%	10.91%	9.09%	5.26%
7		0.00%	29.09%	9.09%	0.00%
8		0.00%	5.45%	0.00%	0.00%
no answer		0.00%	5.45%	6.06%	0.00%
		100.00%	100.00%	100.00%	100.00%

Above is the % withi in each category for each reason

Newton's: Frequency

REASONS	Never	rarely	sometimes	often	always	no reply
1	1	1	1	2	1	
2			13	21	6	
3		4	1	13	8	
4			4	3	1	
5		1	1	1	2	
6			12	5	1	
7			19	10		
8			2	2		
9			2			
no answer		2	1	6		
	1	8	56	63	19	38
						185 TOTAL

Newton's: Percentage

REASONS	Never	rarely	sometimes	often	always
1		12.50%	1.79%	3.77%	5.26%
2		0.00%	23.21%	39.62%	31.58%
3		50.00%	1.79%	24.53%	42.11%
4		0.00%	7.14%	5.66%	5.26%
5		12.50%	1.79%	1.89%	10.53%
6		0.00%	21.43%	9.43%	5.26%
7		0.00%	33.93%	18.87%	0.00%
8		0.00%	3.57%	3.77%	0.00%
		0.00%	3.57%	0.00%	
no answer		25.00%	1.79%	11.32%	0.00%
		100.00%	100.00%	118.87%	100.00%
Above is	the % withi in each		category	for each	reason

Combined L'Hopital's and Newton's: Frequency

REASONS	Never	rarely	sometimes	often	always	no reply
1	2	5	5	5	3	
2		5	28	38	9	
3		7	1	27	17	
4		1	7	13	4	
5		2	6	7	3	
6		0	18	11	2	
7		0	35	16	0	
8		0	5	2	0	
		0	2	0	0	
no answer		2	4	10	0	
	2	22	111	129	38	72
					Total	374

Combined L'Hopital's and Newton's: Percentage

REASONS	Never	rarely	sometimes	often	always
1		22.73%	4.50%	3.88%	7.89%
2		22.73%	25.23%	29.46%	23.68%
3		31.82%	0.90%	20.93%	44.74%
4		4.55%	6.31%	10.08%	10.53%
5		9.09%	5.41%	5.43%	7.89%
6		0.00%	16.22%	8.53%	5.26%
7		0.00%	31.53%	12.40%	0.00%
8		0.00%	4.50%	1.55%	0.00%
		0.00%	1.80%	0.00%	0.00%
no answer		9.09%	3.60%	7.75%	0.00%
		100.00%	100.00%	100.00%	100.00%
Above is	the % withi in each		category	for each	reason

ANALYSIS OF EXTRACT QUESTIONNAIRE PARAGRAPHS A-G
Note: Students could give more than one answer

L'Hopital's

	concentrated on		easiest		hardest		most important	
	N	%	N	%	N	%	N	%
a	26	13.76%	37	19.58%	27	14.29%	47	24.87%
b	84	44.44%	10	5.29%	104	55.03%	123	65.08%
c	23	12.17%	11	5.82%	129	68.25%	43	22.75%
d	133	70.37%	89	47.09%	15	7.94%	103	54.50%
e	139	73.54%	169	89.42%	13	6.88%	85	44.97%
f	54	28.57%	93	49.21%	16	8.47%	32	16.93%
g	27	14.29%	42	22.22%	18	9.52%	11	5.82%

Newton's

	concentrated on		easiest		hardest		most important	
	N	%	N	%	N	%	N	%
a	13	7.03%	81	43.78%	8	4.32%	16	8.65%
b	57	30.81%	38	20.54%	73	39.46%	89	48.11%
c	94	50.81%	25	13.51%	125	67.57%	72	38.92%
d	66	35.68%	23	12.43%	56	30.27%	67	36.22%
e	102	55.14%	63	34.05%	59	31.89%	108	58.38%
f	92	49.73%	156	84.32%	10	5.41%	63	34.05%
g	89	48.11%	104	56.22%	19	10.27%	62	33.51%
h	63	34.05%	68	36.76%	12	6.49%	47	25.41%
D	24	12.97%	44	23.78%	11	5.95%	55	29.73%

PARAGRAPH READ FIRST

L'Hopital's		
a	166	87.83%
b	17	8.99%
c	1	0.53%
d	3	1.59%
e	1	0.53%
f	0	0.00%
g	0	0.00%
no answer	1	0.53%
	189	100.00%

Newton's		
a	160	86.49%
b	2	1.08%
c	2	1.08%
d	0	0.00%
e	9	4.86%
f	1	0.54%
g	0	0.00%
h	0	0.00%
Diagram	7	3.78%
no answer	4	2.16%
	185	100.00%

STUDENT'S OWN OPINION OF THEIR UNDERSTANDING: Students understood the extract

	L'Hopital's		Newton's		Combined	
	N	%	N	%	N	%
1 strongly disagree	2	1.06%	3	1.62%	5	1.34%
2 disagree	6	3.17%	6	3.24%	12	3.21%
3 undecided	42	22.22%	49	26.49%	91	24.33%
4 agree	101	53.44%	106	57.30%	207	55.35%
5 strongly agree	23	12.17%	18	9.73%	41	10.96%
no answer	15	7.94%	3	1.62%	18	4.81%
	189	100.00%	185	100.00%	374	100.00%

OUTPUT OF UNDERSTANDING: 5 QUESTIONS MARKED

	L'Hopital's		Newton's	
	N	%	N	%
1 very little understood	42	22.22%	80	43.24%
2 major ideas but details absent	69	36.51%	54	29.19%
3 major idead and some details	63	33.33%	39	21.08%
4 major ideas, details correct	14	7.41%	7	3.78%
no answer	1	0.53%	5	2.70%
	189	100.00%	185	100.00%
	mean	2.249		1.804
	sd	0.93		0.93
	z			4.62 significantly different

MORE DETAIL ON OUTPUT OF UNDERSTANDING: MARKS OUT OF 10.

	L'Hopital's		Newton's	
	N	%	N	%
0		0.00%	0	0.00%
1	8	4.23%	24	12.97%
2	9	4.76%	13	7.03%
3	19	10.05%	27	14.59%
4	19	10.05%	20	10.81%
5	29	15.34%	37	20.00%
6	43	22.75%	39	21.08%
7	41	21.69%	7	3.78%
8	17	8.99%	10	5.41%
9	3	1.59%	7	3.78%
10	1	0.53%	1	0.54%
	189	100.00%	185	100.00%

PROCEDURE VS CONCEPT VS NEGATIVE VS SYMBOL INTERPRETATION

Newton's: Frequency

	Procedure	Concept	Negative	Symbol
Most or all correct	88	16	86	4
Half correct	67	75	8	39
Incorrect	5	63	21	35
no answer	25	31	70	107
	185	185	185	185

Newton's: Percentages

	Procedure	Concept	Negative	Symbol
Most or all correct	47.57%	8.65%	46.49%	2.16%
Half correct	36.22%	40.54%	4.32%	21.08%
Incorrect	2.70%	34.05%	11.35%	18.92%
no answer	13.51%	16.76%	37.84%	57.84%
	100.00%	100.00%	100.00%	100.00%

*influenced by lack of access to a calculator

L'Hopital's: Frequency

	Procedure	Concept	Negative	Symbol
Most or all correct	142	6	77	37
Half correct	27	103	67	101
Incorrect	5	67	6	20
no answer	15	13	39	31
	189	189	189	189

L'Hopital's: Percentages

	Procedure	Concept	Negative	Symbol
Most or all correct	75.13%	3.17%	40.74%	19.58%
Half correct	14.29%	54.50%	35.45%	53.44%
Incorrect	2.65%	35.45%	3.17%	10.58%
no answer	7.94%	6.88%	20.63%	16.40%
	100.00%	100.00%	100.00%	100.00%

	L'Hopital's	Newton's	Combined	%
Males	137	133	270	72.19%
Females	51	52	103	27.54%
unknown	1		1	0.27%
	189	185	374	100.00%

READ TOPIC BEFOREHAND?

	L'Hopital's		Newton's		Combined	
	N	%	N	%	N	%
YES	22	11.64%	80	43.24%	102	27.27%
NO	154	81.48%	104	56.22%	258	68.98%
no reply	13	6.88%	1	0.54%	14	3.74%
	189	100.00%	185	100.00%	374	100.00%

AGE	L'Hopital's		Newton's		Combined	
<=17	15	7.94%	18	9.73%	33	8.82%
18	101	53.44%	95	51.35%	196	52.41%
19	27	14.29%	32	17.30%	59	15.78%
20	14	7.41%	15	8.11%	29	7.75%
21-25	20	10.58%	17	9.19%	37	9.89%
26-39	7	3.70%	7	3.78%	14	3.74%
>=40	1	0.53%	1	0.54%	2	0.53%
no reply	4	2.12%		0.00%	4	1.07%
	189	1	185	1	374	1

HOW THE STUDENTS READ THE EXTRACT

	L'Hopital's		Newton's	
1= Skim read once only before answering	42	22.22%	38	20.54%
2= Read once slowly, (often rereading each)	28	14.81%	20	10.81%
3= Skim once,then to part not understood	60	31.75%	52	28.11%
4= Skim once, then to examples	20	10.58%	34	18.38%
5=Skim once then to diagrams	1	0.53%	2	1.08%
6=Twice, once skim and 2nd slowly	9	4.76%	6	3.24%
7=Conc on details and introd, then skim rest	7	3.70%	11	5.95%
8=Mainly on Diagram (or steps) then skim rest	2	1.06%	3	1.62%
9=Concentrated on examples then skim rest	3	1.59%	2	1.08%
10=concentrated on definition then skim rese		0.00%	1	0.54%
11= Skim read several times	13	6.88%	0	0.00%
no answer	4	2.12%	16	8.65%
	189	100.00%	185	100.00%

	Combined	
1= Skim read once only before answering		
2= Read once slowly, (often rereading each)	80	21.39%
3= Skim once,then to part not understood	48	12.83%
4= Skim once, then to examples	112	29.95%
5=Skim once then to diagrams	54	14.44%
6=Twice, once skim and 2nd slowly	3	0.80%
7=Conc on details and introd, then skim rest	15	4.01%
8=Mainly on Diagram (or steps) then skim rest	18	4.81%
9=Concentrated on examples then skim rest	5	1.34%
10=concentrated on definition then skim rese	5	1.34%
11= Skim read several times	1	0.27%
no answer	13	3.48%
	20	5.35%
	374	100.00%

UNDERSTANDING THE EXTRACT:
OWN OPINION VS ACTUAL OUTPUT FROM MARKS.
Correlation Coefficients

L'Hopital's

	output/10	*own*
output/10	1	
own	0.409686	1

	own	*output/4*
own	1	
output/4	0.397332	1

UNDERSTANDING THE EXTRACT:
OWN OPINION VS ACTUAL OUTPUT FROM MARKS.
Correlation Coefficents

Newton's

	output/10	*own*
output/10	1	
own	0.282634	1

	own	*output/4*
own	1	
output/4	0.261032	1

COMPARISON OF HOW EXTRACT WAS READ vs MARK GAINED
L'Hopital's: How extract was read vs mark/4:

Type 1:		Type 2		Type 3		Type 4	
Mean mark/·	1.790698	Mean	2.9642857	Mean	2.360656	Mean	2.238095
Standard Er	0.108131	Standard E	0.1583826	Standard E	0.114598	Standard E	0.13636
Median	2	Median	3	Median	2	Median	2
Mode	2	Mode	3	Mode	3	Mode	2
Standard De	0.709062	Standard D	0.8380817	Standard D	0.895038	Standard D	0.624881
Sample Vari	0.502769	Sample Va	0.702381	Sample Va	0.801093	Sample Va	0.390476
Kurtosis	-0.916276	Kurtosis	0.5075924	Kurtosis	-0.317246	Kurtosis	-0.364914
Skewness	0.325947	Skewness	-0.743093	Skewness	-0.357293	Skewness	-0.195159
Range	2	Range	3	Range	4	Range	2
Minimum	1	Minimum	1	Minimum	0	Minimum	1
Maximum	3	Maximum	4	Maximum	4	Maximum	3
Sum	77	Sum	83	Sum	144	Sum	47
Count	43	Count	28	Count	61	Count	21
Confidence	0.218217	Confidence	0.3249739	Confidence	0.22923	Confidence	0.284442

(95% CI)

COMPARISON OF HOW EXTRACT WAS READ vs MARK GAINED
L'Hopital's: How extract was read vs mark/4:

Type 6		Type 7		Type 11	
Mean	2.8	Mean	2.8571429	Mean	1.230769
Standard Er	0.249444	Standard E	0.2608203	Standard E	0.121626
Median	3	Median	3	Median	1
Mode	2	Mode	3	Mode	1
Standard De	0.788811	Standard D	0.6900656	Standard D	0.438529
Sample Vari	0.622222	Sample Va	0.4761905	Sample Va	0.192308
Kurtosis	-1.074162	Kurtosis	0.336	Kurtosis	0.094545
Skewness	0.407485	Skewness	0.1738965	Skewness	1.451132
Range	2	Range	2	Range	1
Minimum	2	Minimum	2	Minimum	1
Maximum	4	Maximum	4	Maximum	2
Sum	28	Sum	20	Sum	16
Count	10	Count	7	Count	13
Confidence	0.564282	Confidence	0.6382047	Confidence	0.265

Newton's
Type 5: 2 students, marks 1 and 3.
Type 9:, 2 students, marks 1 and 2.
Type 10, 1 student, marks is 2.

L'Hopital's
Type 5 only had one student: Mark was 2
Type 8 only had two students: Both marks were 2.
Type 9 only two students: Marks, 1 and 2.
No type 10.

Newton's

Type 1		Type 2		Type 3		Type 4	
Mean	1.5	Mean	2.3	Mean	2.038462	Mean	1.852941
Standard Er	0.154358	Standard E	0.2064742	Standard E	0.128694	Standard E	0.152999
Median	1	Median	2.5	Median	2	Median	2
Mode	1	Mode	3	Mode	2	Mode	1
Standard De	0.951528	Standard D	0.9233805	Standard D	0.928028	Standard D	0.892132
Sample Vari	0.905405	Sample Va	0.8526316	Sample Va	0.861237	Sample Va	0.7959
Kurtosis	1.020641	Kurtosis	-1.071483	Kurtosis	-0.501536	Kurtosis	-0.823235
Skewness	0.993427	Skewness	-0.231748	Skewness	0.227815	Skewness	0.575674
Range	4	Range	3	Range	4	Range	3
Minimum	0	Minimum	1	Minimum	0	Minimum	1
Maximum	4	Maximum	4	Maximum	4	Maximum	4
Sum	57	Sum	46	Sum	106	Sum	63
Count	38	Count	20	Count	52	Count	34
Confidence	0.312759	Confidence	0.4321555	Confidence	0.258365	Confidence	0.31128

Type 6		Type 7		Type 8	
Mean	1.333333	Mean	1.8181818	Mean	1.333333
Standard Er	0.421637	Standard E	0.2634796	Standard E	0.333333
Median	1	Median	2	Median	1
Mode	1	Mode	1	Mode	1
Standard De	1.032796	Standard D	0.8738629	Standard D	0.57735
Sample Vari	1.066667	Sample Va	0.7636364	Sample Va	0.333333
Kurtosis	0.585937	Kurtosis	-1.621315	Kurtosis	#DIV/0!
Skewness	0.665669	Skewness	0.4086944	Skewness	1.732051
Range	3	Range	2	Range	1
Minimum	0	Minimum	1	Minimum	1
Maximum	3	Maximum	3	Maximum	2
Sum	8	Sum	20	Sum	4
Count	6	Count	11	Count	3
Confidence	1.083851	Confidence	0.5870692	Confidence	1.434219

SIGNIFICANT DIFFERENCES?

	number	mean	s.d
1	80	1.6	0.8
2	15	2.1	0.96
3	112	2.1	0.91
4	54	2	0.76
5	48	2.7	0.87
6	18	2.3	0.78

ns = not significant
s = significant

		t
1&2	ns	1.9
1&3	s	4.03
1&4	s	2.94
1&5	s	7.14
1&6	s	3.42
2&3	ns	0
2&4	ns	1.48
2&5	s	2.17
2&6	ns	0.64
3&4	ns	0.74
3&5	s	3.8
3&6	ns	0.98
4&5	s	4.3
4&6	ns	1.42
5&6	ns	1.79

APPENDIX F

DATA FOR A COMPARISON OF LEARNING RESOURCES (CHAPTER 8)

TEXTBOOK
CD-TEXT
EPSILON
MULTIMEDIA

t-test values for a comparison between technology and textbook studies

The value in brackets is the critical value for t for a two-tailed test.

	CD-text pairs	Epsilon pairs	CD-text individual	Epsilon individual	Textbook pairs	Textbook individual	Multimedia individual	Multimedia pairs
CD-text Pairs	1							
Epsilon Pairs	0.69 (1.99)	1						
CD-Text Individual	0.87 (1.98)	1.11 (2.01)	1					
Epsilon Individual	1.54 (2.00)	1.33 (2.01)	1.12 (1.98)	1				
Textbook Pairs	4.07 * (1.99)	3.32 * (1.99)	4.42* (1.99)	1.54 (2.00)	1			
Textbook Individual	3.17 * (2.03)	3.75 * (1.97)	3.17 * (2.03)	2.10 (2.01)	0.94 (2.01)	1		
Multimedia Individual	8.59 * (1.99)	7.99 * (1.98)	5.16* (1.97)	3.42 * (2.00)	3.32 * (1.97)	0.61 (2.01)	1	
Multimedia Pairs	9.36 * (2.01)	8.27 * (2.00)	8.50 * (1.96)	7.99 * (1.98)	4.22 * (1.98)	3.18* (1.97)	2.15 * (1.99)	1

* means that items are significantly different

t-test values for a comparison between pretest and posttest

Pretest

Posttest

Posttest ↓ / Pretest	CD-text pairs	Epsilon pairs	CD-text individual	Epsilon individual	Textbook pairs	Textbook individual	Multimedia individual	Multimedia pairs
CD-text Pairs	24.09 * (2.03)							
Epsilon Pairs		19.31 * (2.03)						
CD-Text Individual			15.87 * (2.03)					
Epsilon Individual				14.10* (2.07)				
Textbook Pairs					10.24* (2.01)			
Textbook Individual						5.83 * (2.07)		
Multimedia Individual							7.27 * (2.02)	
Multimedia Pairs								7.51 * (2.03)

* All posttest scores are significantly different from pretest scores.